

# Spinor Structure and Modulo 8 Periodicity

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## Abstract

Spinor structure is understood as a totality of tensor products of biquaternion algebras, and the each tensor product is associated with an irreducible representation of the Lorentz group. A so-defined algebraic structure allows one to apply modulo 8 periodicity of Clifford algebras on the system of real and quaternionic representations of the Lorentz group. It is shown that modulo 8 periodic action of the Brauer-Wall group generates modulo 2 periodic relations on the system of representations, and all the totality of representations under this action forms a self-similar fractal structure. Some relations between spinors, twistors and qubits are discussed in the context of quantum information and decoherence theory.

**Keywords:** spinor structure, Clifford algebras, spinors, twistors, qubits

## 1 Introduction

A geometric description of space-time continuum via the representation of time by an imaginary coordinate of a four-dimensional pseudo-Euclidean space, which was first given by Minkowski, can be considered as an original point for subsequent programme of *geometrization* of physics initiated in the works of Poincaré, Einstein, Klein (theories of special and general relativity), Weyl, Kaluza, Eddington (multidimensional generalizations of relativistic theories, unification of gravitation and electromagnetism), Wheeler, Connes and other (geometrodynamics, noncommutative geometry and so on). On the other hand, along with the geometry algebraic methods, revealing the unity of mathematics, were penetrated into physics, first of all, methods of group theory (Lorentz and Poincaré groups, Erlangen programme, group-theoretical description of quantum mechanics and so on). In such a way, the programme of geometrization of physics involves a programme of *algebraization* of physics. In this programme Clifford algebras and theory of hypercomplex structures as a whole play an essential role. A wide application of Clifford algebras and spinors in physics began with the famous Dirac's work on electron theory [1], where well-known  $\gamma$ -matrices form a basis of the complex Clifford algebra  $\mathbb{C}_4$ . As is known, a fundamental notion of *antimatter* follows directly from the Dirac equation which presents by itself a *spinor equation*. Later on, Laport and Uhlenbeck [2], using van der Waerden 2-spinor formalism [3], wrote Maxwell equations in a *spinor form*. It allows one to consider all the relativistic wave equations on an equal footing, from the one algebraic (group-theoretical) viewpoint [4]. Clifford algebras form the foundation of twistor approach. In accordance with Penrose twistor programme [5, 6], space-time continuum is a derivative construction with respect to *underlying spinor structure*. Spinor structure contains in itself pre-images of all basic properties of classical space-time, such as dimension, signature, metrics and many other. In parallel with twistor approach, decoherence theory [7] claims that in the background of reality we have a *nonlocal quantum substrate* (quantum domain), and all visible world (classical domain) arises from quantum domain in the result of decoherence process. In this

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context Clifford algebras should be understood as a mathematical tool working on the level of quantum domain.

The present paper is a continuation of the previous work [8]. In this paper we study applications of modulo 8 periodicity of Clifford algebras to particle representations of the group  $\mathbf{Spin}_+(1, 3)$  in more details. The underlying spinor structure is understood here as an algebraic structure (tensor products of Clifford algebras) associated with the each finite-dimensional representation of  $\mathbf{Spin}_+(1, 3)$ . In accordance with the Wigner interpretation of elementary particles [9], a so-defined spinor structure contains pre-images of all basic properties of elementary particles, such as spin, mass [10], charge (a pseudoautomorphism  $\mathcal{A} \rightarrow \overline{\mathcal{A}}$ , [11, 12]), space inversion  $P$ , time reversal  $T$  and their combination  $PT$  (fundamental automorphisms  $\mathcal{A} \rightarrow \mathcal{A}^*$ ,  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ ,  $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}^*$  of Clifford algebras [13, 14]). It is shown that modulo 8 periodicity of underlying spinor structure generates modulo 2 periodic relations on the system of representations (real and quaternionic) of the group  $\mathbf{Spin}_+(1, 3)$ . In consequence of the action of the Brauer-Wall group  $BW_{\mathbb{R}}$  we have a self-similar fractal structure on the system of representations of  $\mathbf{Spin}_+(1, 3)$ , where the each period of this fractal structure is generated by the cycle of  $BW_{\mathbb{R}}$ . It is shown also that modulo 8 action of  $BW_{\mathbb{R}}$  induces modulo 4 periodic relations on the idempotents groups of the Clifford algebras. Some relations between spinors, twistors and qubits are discussed with respect to theory of quantum information.

## 2 Periodicity of Clifford algebras

As is known, for the Clifford algebra  $\mathcal{C}_{p,q}$  over the field  $\mathbb{F} = \mathbb{R}$  there are isomorphisms  $\mathcal{C}_{p,q} \simeq \text{End}_{\mathbb{K}}(I_{p,q}) \simeq \text{Mat}_{2^m}(\mathbb{K})$ , where  $m = (p + q)/2$ ,  $I_{p,q} = \mathcal{C}_{p,q}f$  is a minimal left ideal of  $\mathcal{C}_{p,q}$ , and  $\mathbb{K} = f\mathcal{C}_{p,q}f$  is a division ring of  $\mathcal{C}_{p,q}$ . A primitive idempotent of the algebra  $\mathcal{C}_{p,q}$  has the form

$$f = \frac{1}{2}(1 \pm \mathbf{e}_{\alpha_1}) \frac{1}{2}(1 \pm \mathbf{e}_{\alpha_2}) \cdots \frac{1}{2}(1 \pm \mathbf{e}_{\alpha_k}),$$

where  $\mathbf{e}_{\alpha_1}, \mathbf{e}_{\alpha_2}, \dots, \mathbf{e}_{\alpha_k}$  are commuting elements with square 1 of the canonical basis of  $\mathcal{C}_{p,q}$  generating a group of order  $2^k$ , that is,  $(\mathbf{e}_{\alpha_1}, \mathbf{e}_{\alpha_2}, \dots, \mathbf{e}_{\alpha_k}) \simeq (\mathbb{Z}_2)^{\otimes k}$ , where  $(\mathbb{Z}_2)^{\otimes k} = \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \cdots \otimes \mathbb{Z}_2$  ( $k$  times) is an Abelian group. The values of  $k$  are defined by the formula  $k = q - r_{q-p}$ , where  $r_i$  are the Radon-Hurwitz numbers [15, 16], values of which form a cycle of the period 8:  $r_{i+8} = r_i + 4$ . The values of all  $r_i$  are

$$\begin{array}{c|cccccccc} i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline r_i & 0 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \end{array}.$$

In terms of finite groups we have here *an idempotent group*  $T_{p,q}(f) \simeq (\mathbb{Z}_2)^{\otimes(k+1)}$  of the order  $2^{k+1} = 2^{1+q-r_{q-p}}$  [17].

For example, let us consider a minimal left ideal of the de Sitter algebra  $\mathcal{C}_{4,1}$  associated with the space  $\mathbb{R}^{4,1}$ . This ideal has the form

$$I_{4,1} = \mathcal{C}_{4,1}f_{4,1} = \mathcal{C}_{4,1}\frac{1}{2}(1 + \mathbf{e}_0)\frac{1}{2}(1 + i\mathbf{e}_{12}). \quad (1)$$

In its turn, for the space-time algebra  $\mathcal{C}_{1,3}$  we have

$$I_{1,3} = \mathcal{C}_{1,3}f_{13} = \mathcal{C}_{1,3}\frac{1}{2}(1 + \mathbf{e}_0).$$

Further, for the Dirac algebra there are isomorphisms  $\mathcal{C}_{4,1} \simeq \mathbb{C}_4 = \mathbb{C} \otimes \mathcal{C}_{1,3} \simeq \text{Mat}_2(\mathbb{C}_2)$ ,  $\mathcal{C}_{4,1}^+ \simeq \mathcal{C}_{1,3} \simeq \text{Mat}_2^{\mathbb{H}}(\mathcal{C}_{1,1})$ . Using an identity  $\mathcal{C}_{1,3}f_{13} = \mathcal{C}_{1,3}^+f_{13}$  [18], we obtain for the minimal

left ideal of  $\mathcal{C}_{4,1}$  the following expression:

$$I_{4,1} = \mathcal{C}_{4,1}f_{41} = (\mathbb{C} \otimes \mathcal{C}_{1,3})f_{41} \simeq \mathcal{C}_{4,1}^+f_{41} \simeq \mathcal{C}_{1,3}f_{41} = \mathcal{C}_{1,3}f_{13}\frac{1}{2}(1 + i\mathbf{e}_{12}) = \mathcal{C}_{1,3}^+f_{13}\frac{1}{2}(1 + i\mathbf{e}_{12}). \quad (2)$$

Let  $\Phi \in \mathcal{C}_{4,1} \simeq \text{Mat}_4(\mathbb{C})$  be a Dirac spinor and let  $\phi \in \mathcal{C}_{1,3}^+ \simeq \mathcal{C}_{3,0}$  be a Dirac-Hestenes spinor. Then from (2) we have the relation between spinors  $\Phi$  and  $\phi$ :

$$\Phi = \phi\frac{1}{2}(1 + \mathbf{e}_0)\frac{1}{2}(1 + i\mathbf{e}_{12}). \quad (3)$$

Since  $\phi \in \mathcal{C}_{1,3}^+ \simeq \mathcal{C}_{3,0}$ , the Dirac-Hestenes spinor can be represented by a biquaternion number

$$\phi = a^0 + a^{01}\mathbf{e}_{01} + a^{02}\mathbf{e}_{02} + a^{03}\mathbf{e}_{03} + a^{12}\mathbf{e}_{12} + a^{13}\mathbf{e}_{13} + a^{23}\mathbf{e}_{23} + a^{0123}\mathbf{e}_{0123}. \quad (4)$$

It should be noted that Hestenes programme of reinterpretation of quantum mechanics within the field of real numbers gives rise many interesting applications of Dirac-Hestenes fields in geometry and general theory of relativity [19, 20, 21, 22, 23].

Over the field  $\mathbb{F} = \mathbb{R}$  there are eight different types of Clifford algebras  $\mathcal{C}_{p,q}$  with the following division ring structure.

#### I. Central simple algebras.

1. Two types  $p - q \equiv 0, 2 \pmod{8}$  with a division ring  $\mathbb{K} \simeq \mathbb{R}$ .
2. Two types  $p - q \equiv 3, 7 \pmod{8}$  with a division ring  $\mathbb{K} \simeq \mathbb{C}$ .
3. Two types  $p - q \equiv 4, 6 \pmod{8}$  with a division ring  $\mathbb{K} \simeq \mathbb{H}$ .

#### II. Semi-simple algebras.

4. The type  $p - q \equiv 1 \pmod{8}$  with a double division ring  $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$ .
5. The type  $p - q \equiv 5 \pmod{8}$  with a double quaternionic division ring  $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$ .

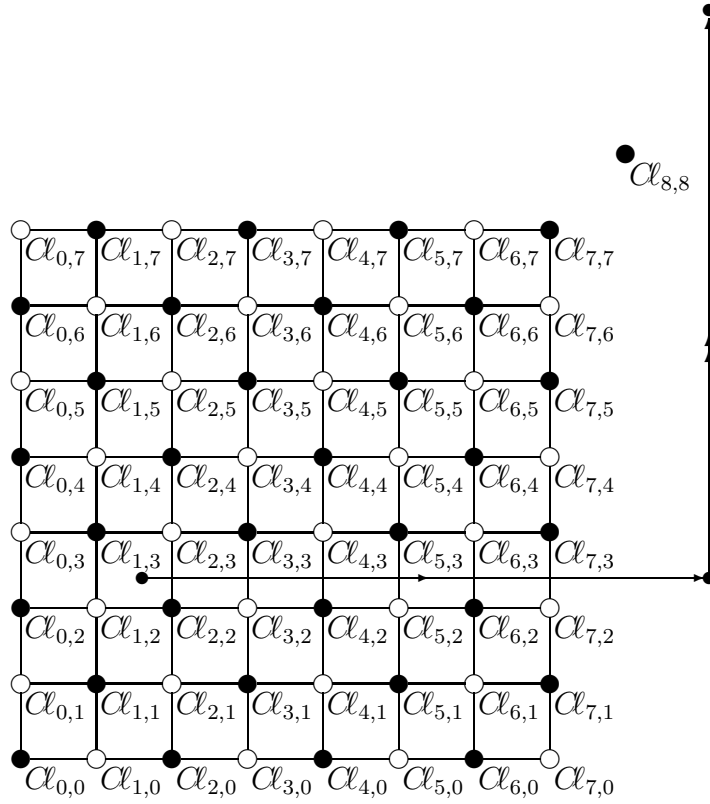
The *spinorial chessboard* [24] (see Fig. 1) is the set of 64 real algebras

$$\{\mathcal{C}_{p,q}, \mid 0 \leq p, q \leq 7\},$$

where it is understood that  $\mathcal{C}_{0,0} \simeq \mathbb{R}$ .

The algebra  $\mathcal{C}$  is naturally  $\mathbb{Z}_2$ -graded. Let  $\mathcal{C}^+$  (correspondingly  $\mathcal{C}^-$ ) be a set consisting of all even (correspondingly odd) elements of the algebra  $\mathcal{C}$ . The set  $\mathcal{C}^+$  is a subalgebra of  $\mathcal{C}$ . It is obvious that  $\mathcal{C} = \mathcal{C}^+ \oplus \mathcal{C}^-$ , and also  $\mathcal{C}^+\mathcal{C}^+ \subset \mathcal{C}^+$ ,  $\mathcal{C}^+\mathcal{C}^- \subset \mathcal{C}^-$ ,  $\mathcal{C}^-\mathcal{C}^+ \subset \mathcal{C}^-$ ,  $\mathcal{C}^-\mathcal{C}^- \subset \mathcal{C}^+$ . A degree  $\deg a$  of the even (correspondingly odd) element  $a \in \mathcal{C}$  is equal to 0 (correspondingly 1). Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be the two associative  $\mathbb{Z}_2$ -graded algebras over the field  $\mathbb{F}$ ; then a multiplication of homogeneous elements  $\mathbf{a}' \in \mathfrak{A}$  and  $\mathbf{b} \in \mathfrak{B}$  in a graded tensor product  $\mathfrak{A} \hat{\otimes} \mathfrak{B}$  is defined as follows:  $(\mathbf{a} \otimes \mathbf{b})(\mathbf{a}' \otimes \mathbf{b}') = (-1)^{\deg \mathbf{b} \deg \mathbf{a}'} \mathbf{a}\mathbf{a}' \otimes \mathbf{b}\mathbf{b}'$ .

**Theorem 1** (Chevalley [25]). *Let  $V$  and  $V'$  are vector spaces over the field  $\mathbb{F}$  and let  $Q$  and  $Q'$  are quadratic forms for  $V$  and  $V'$ . Then a Clifford algebra  $\mathcal{C}(V \oplus V', Q \oplus Q')$  is naturally isomorphic to  $\mathcal{C}(V, Q) \hat{\otimes} \mathcal{C}(V', Q')$ .*



**Fig. 1: The Spinorial Chessboard.** Even- and odd-dimensional Clifford algebras  $\mathcal{C}_{p,q}$ ,  $0 \leq p, q \leq 7$ , occupy, respectively, black and white circles (squares of the board). Every real Clifford algebra can be reached from one on the board with rook's moves to the right and upward.

Let  $\mathcal{C}(V, Q)$  be the Clifford algebra over the field  $\mathbb{F} = \mathbb{R}$ , where  $V$  is a vector space endowed with quadratic form  $Q = x_1^2 + \dots + x_p^2 - \dots - x_{p+q}^2$ . If  $p + q$  is even and  $\omega^2 = 1$ , then  $\mathcal{C}(V, Q)$  is called *positive* and correspondingly *negative* if  $\omega^2 = -1$ , that is,  $\mathcal{C}_{p,q} > 0$  if  $p - q \equiv 0, 4 \pmod{8}$  and  $\mathcal{C}_{p,q} < 0$  if  $p - q \equiv 2, 6 \pmod{8}$ .

**Theorem 2** (Karoubi [26, Prop. 3.16]). 1) If  $\mathcal{C}(V, Q) > 0$  and  $\dim V$  is even, then

$$\mathcal{C}(V \oplus V', Q \oplus Q') \simeq \mathcal{C}(V, Q) \otimes \mathcal{C}(V', Q').$$

2) If  $\mathcal{C}(V, Q) < 0$  and  $\dim V$  is even, then

$$\mathcal{C}(V \oplus V', Q \oplus Q') \simeq \mathcal{C}(V, Q) \otimes \mathcal{C}(V', -Q').$$

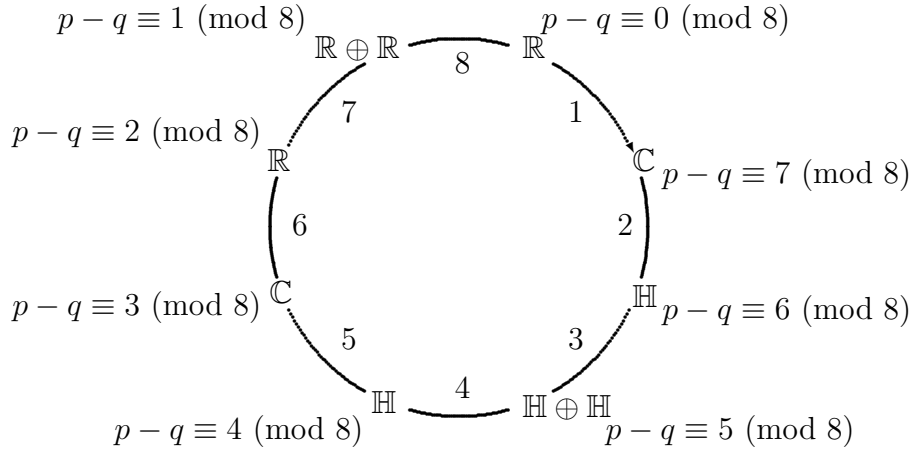
Over the field  $\mathbb{F} = \mathbb{C}$  the Clifford algebra is always positive (if  $\omega^2 = \mathbf{e}_{12\dots p+q}^2 = -1$ , then we can suppose  $\omega = i\mathbf{e}_{12\dots p+q}$ ). Thus, using Karoubi Theorem, we find that

$$\underbrace{\mathbb{C}_2 \otimes \dots \otimes \mathbb{C}_2}_{m \text{ times}} \simeq \mathbb{C}_{2m}. \quad (5)$$

Therefore, the tensor product in (5) is isomorphic to  $\mathbb{C}_{2m}$ . For example, there are two different factorizations  $\mathcal{C}_{1,1} \otimes \mathcal{C}_{0,2}$  and  $\mathcal{C}_{1,1} \otimes \mathcal{C}_{2,0}$  for the spacetime algebra  $\mathcal{C}_{1,3}$  and Majorana algebra  $\mathcal{C}_{3,1}$ .

The real Clifford algebra  $\mathcal{C}_{p,q}$  is central simple if  $p - q \not\equiv 1, 5 \pmod{8}$ . The graded tensor product of the two graded central simple algebras is also graded central simple [27, Theorem

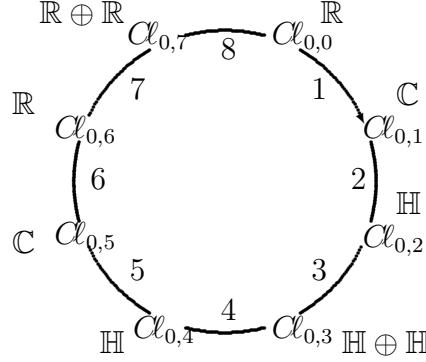
2]. It is known that for the Clifford algebra with odd dimensionality, the isomorphisms are as follows:  $\mathcal{C}_{p,q+1}^+ \simeq \mathcal{C}_{p,q}$  and  $\mathcal{C}_{p+1,q}^+ \simeq \mathcal{C}_{q,p}$  [28, 29]. Thus,  $\mathcal{C}_{p,q+1}^+$  and  $\mathcal{C}_{p+1,q}^+$  are central simple algebras. Further, in accordance with Chevalley Theorem for the graded tensor product there is an isomorphism  $\mathcal{C}_{p,q} \hat{\otimes} \mathcal{C}_{p',q'} \simeq \mathcal{C}_{p+p',q+q'}$ . Two algebras  $\mathcal{C}_{p,q}$  and  $\mathcal{C}_{p',q'}$  are said to be of the same class if  $p + q' \equiv p' + q \pmod{8}$ . The graded central simple Clifford algebras over the field  $\mathbb{F} = \mathbb{R}$  form eight similarity classes, which, as it is easy to see, coincide with the eight types of the algebras  $\mathcal{C}_{p,q}$ . The set of these 8 types (classes) forms a Brauer-Wall group  $BW_{\mathbb{R}}$  [27, 30]. It is obvious that an action of  $BW_{\mathbb{R}}$  has a cyclic structure, which is formally equivalent to the action of cyclic group  $\mathbb{Z}_8$ . The cyclic structure of  $BW_{\mathbb{R}}$  may be represented on the Budinich-Trautman diagram (spinorial clock) [24] (Fig. 2) by means of a transition  $\mathcal{C}_{p,q}^+ \xrightarrow{h} \mathcal{C}_{p,q}$  (the round on the diagram is realized by an hour-hand). At this point, the type of the algebra is defined on the diagram by an equality  $q - p = h + 8r$ , where  $h \in \{1, \dots, 8\}$ ,  $r \in \mathbb{Z}$ . It is obvious that a group structure over  $\mathcal{C}_{p,q}$ , defined by  $BW_{\mathbb{R}}$ , is related with the Atiyah-Bott-Shapiro periodicity [31].



**Fig. 2: The Spinorial Clock.** The Budinich-Trautman diagram for the Brauer-Wall group  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$ .

On the other hand, graded central simple Clifford algebras over the field  $\mathbb{F} = \mathbb{R}$  form a **graded Brauer group**  $G(\mathcal{C}_{p,q}, \gamma, \odot)$  [27, 8], a cyclic structure of which is described by the Brauer-Wall group  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$  [30]. Therefore, a cyclic structure of  $G(\mathcal{C}_{p,q}, \gamma, \odot) \sim BW_{\mathbb{R}}$  is defined by a transition  $\mathcal{C}_{p,q}^+ \xrightarrow{h} \mathcal{C}_{p,q}$ , where the type of  $\mathcal{C}_{p,q}$  is defined by the formula  $q - p = h + 8r$ , here  $h \in \{1, \dots, 8\}$ ,  $r \in \mathbb{Z}$  [24]. Let us consider in detail several action cycles of  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$ . In virtue of an isomorphism  $\mathcal{C}_{0,1}^+ \simeq \mathcal{C}_{0,0}$  a transition  $\mathcal{C}_{0,1}^+ \xrightarrow{1} \mathcal{C}_{0,1}$  leads to a transition  $\mathcal{C}_{0,0} \xrightarrow{1} \mathcal{C}_{0,1}$ , that is,  $\mathbb{R} \xrightarrow{1} \mathbb{C}$ . At this point,  $h = 1$  and  $r = 0$  (an original point of the first cycle). Further, in virtue of  $\mathcal{C}_{0,2}^+ \simeq \mathcal{C}_{0,1}$  a transition  $\mathcal{C}_{0,2}^+ \xrightarrow{2} \mathcal{C}_{0,2}$  induces a transition  $\mathcal{C}_{0,1} \xrightarrow{2} \mathcal{C}_{0,2}$  ( $\mathbb{C} \xrightarrow{2} \mathbb{H}$ ), at this point,  $h = 2$  and  $r = 0$ . The following transition  $\mathcal{C}_{0,3}^+ \xrightarrow{3} \mathcal{C}_{0,3}$  ( $\mathbb{H} \xrightarrow{3} \mathbb{H} \oplus \mathbb{H}$ ) in virtue of  $\mathcal{C}_{0,3}^+ \simeq \mathcal{C}_{0,2}$  leads to a transition  $\mathcal{C}_{0,2} \xrightarrow{3} \mathcal{C}_{0,3}$ . At this transition we have  $h = 3$  and  $r = 0$ . In virtue of an isomorphism  $\mathcal{C}_{0,4}^+ \simeq \mathcal{C}_{0,3}$  a transition  $\mathcal{C}_{0,4}^+ \xrightarrow{4} \mathcal{C}_{0,4}$  ( $\mathbb{H} \oplus \mathbb{H} \xrightarrow{4} \mathbb{H}$ ) induces  $\mathcal{C}_{0,3} \xrightarrow{4} \mathcal{C}_{0,4}$ , at this point,  $h = 4$  and  $r = 0$ . Further, in virtue of  $\mathcal{C}_{0,5}^+ \simeq \mathcal{C}_{0,4}$  a transition  $\mathcal{C}_{0,5}^+ \xrightarrow{5} \mathcal{C}_{0,5}$  ( $\mathbb{H} \xrightarrow{5} \mathbb{C}$ ) induces  $\mathcal{C}_{0,4} \xrightarrow{5} \mathcal{C}_{0,5}$ . At this transition we have  $h = 5$  and  $r = 0$ . The following transition  $\mathcal{C}_{0,6}^+ \xrightarrow{6} \mathcal{C}_{0,6}$  ( $\mathbb{C} \xrightarrow{6} \mathbb{R}$ ) in virtue of  $\mathcal{C}_{0,6}^+ \simeq \mathcal{C}_{0,5}$  induces  $\mathcal{C}_{0,5} \xrightarrow{6} \mathcal{C}_{0,6}$ , here  $h = 6$  and  $r = 0$ . In its turn, the transition  $\mathcal{C}_{0,7}^+ \xrightarrow{7} \mathcal{C}_{0,7}$  ( $\mathbb{R} \xrightarrow{7} \mathbb{R} \oplus \mathbb{R}$ ) in virtue of

$\mathcal{A}_{0,7}^+ \simeq \mathcal{A}_{0,6}$  induces  $\mathcal{A}_{0,6} \xrightarrow{7} \mathcal{A}_{0,7}$ . At this transition we have  $h = 7$  and  $r = 0$ . Finally, a transition  $\mathcal{A}_{0,8}^+ \xrightarrow{8} \mathcal{A}_{0,8}$  ( $\mathbb{R} \oplus \mathbb{R} \xrightarrow{8} \mathbb{R}$ ) finishes the first cycle ( $h = 8, r = 0$ ) and in virtue of  $\mathcal{A}_{0,8}^+ \simeq \mathcal{A}_{0,7}$  induces the following transition  $\mathcal{A}_{0,7} \xrightarrow{8} \mathcal{A}_{0,8}$ . The full round of the first cycle is shown on the Fig. 3. The first cycle generates the first eight squares ( $\mathcal{A}_{0,q}, q = 0, \dots, 7$ ) of the spinorial chessboard (see Fig. 1). The following eight squares ( $\mathcal{A}_{1,q}, q = 0, \dots, 7$ ) are generated by the first cycle also (via the rule  $\mathcal{A}_{1,q} \simeq \mathcal{A}_{1,0} \otimes \mathcal{A}_{0,q}, q = 0, \dots, 7$ ) and so on ( $\mathcal{A}_{2,q} \simeq \mathcal{A}_{2,0} \otimes \mathcal{A}_{0,q}, \mathcal{A}_{3,q} \simeq \mathcal{A}_{3,0} \otimes \mathcal{A}_{0,q}, \dots, \mathcal{A}_{6,q} \simeq \mathcal{A}_{6,0} \otimes \mathcal{A}_{0,q}, q = 0, \dots, 7$ ). In like manner all the squares of the spinorial chessboard, shown on the Fig. 1, are filled.



**Fig. 3:** The first cycle of  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$ .

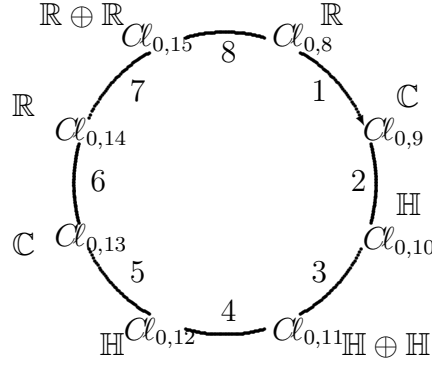
The second cycle ( $h = 1, r = 1$ ) is started with a transition  $\mathcal{A}_{0,9}^+ \xrightarrow{1} \mathcal{A}_{0,9}$  ( $\mathcal{A}_{0,8} \xrightarrow{1} \mathcal{A}_{0,9}$ ) and so on:

- 1)  $\mathcal{A}_{0,9}^+ \xrightarrow{1} \mathcal{A}_{0,9}$  ( $\mathcal{A}_{0,8} \xrightarrow{1} \mathcal{A}_{0,9}$ ),  $h = 1, r = 1, \mathbb{R} \xrightarrow{1} \mathbb{C}$ ;
- 2)  $\mathcal{A}_{0,10}^+ \xrightarrow{2} \mathcal{A}_{0,10}$  ( $\mathcal{A}_{0,9} \xrightarrow{2} \mathcal{A}_{0,10}$ ),  $h = 2, r = 1, \mathbb{C} \xrightarrow{2} \mathbb{H}$ ;
- 3)  $\mathcal{A}_{0,11}^+ \xrightarrow{3} \mathcal{A}_{0,11}$  ( $\mathcal{A}_{0,10} \xrightarrow{3} \mathcal{A}_{0,11}$ ),  $h = 3, r = 1, \mathbb{H} \xrightarrow{3} \mathbb{H} \oplus \mathbb{H}$ ;
- 4)  $\mathcal{A}_{0,12}^+ \xrightarrow{4} \mathcal{A}_{0,12}$  ( $\mathcal{A}_{0,11} \xrightarrow{4} \mathcal{A}_{0,12}$ ),  $h = 4, r = 1, \mathbb{H} \oplus \mathbb{H} \xrightarrow{4} \mathbb{H}$ ;
- 5)  $\mathcal{A}_{0,13}^+ \xrightarrow{5} \mathcal{A}_{0,13}$  ( $\mathcal{A}_{0,12} \xrightarrow{5} \mathcal{A}_{0,13}$ ),  $h = 5, r = 1, \mathbb{H} \xrightarrow{5} \mathbb{C}$ ;
- 6)  $\mathcal{A}_{0,14}^+ \xrightarrow{6} \mathcal{A}_{0,14}$  ( $\mathcal{A}_{0,13} \xrightarrow{6} \mathcal{A}_{0,14}$ ),  $h = 6, r = 1, \mathbb{C} \xrightarrow{6} \mathbb{R}$ ;
- 7)  $\mathcal{A}_{0,15}^+ \xrightarrow{7} \mathcal{A}_{0,15}$  ( $\mathcal{A}_{0,14} \xrightarrow{7} \mathcal{A}_{0,15}$ ),  $h = 7, r = 1, \mathbb{R} \xrightarrow{7} \mathbb{R} \oplus \mathbb{R}$ ;
- 8)  $\mathcal{A}_{0,16}^+ \xrightarrow{8} \mathcal{A}_{0,16}$  ( $\mathcal{A}_{0,15} \xrightarrow{8} \mathcal{A}_{0,16}$ ),  $h = 8, r = 1, \mathbb{R} \oplus \mathbb{R} \xrightarrow{8} \mathbb{R}$ .

The full round of the second cycle is shown on the Fig. 4.

Further, the eighth cycle ( $r = 7$ ) finishes the construction of eight squares of the new spinorial chessboard (fractal self-similar algebraic structure of the second order):

- 1)  $\mathcal{A}_{0,57}^+ \xrightarrow{1} \mathcal{A}_{0,57}$  ( $\mathcal{A}_{0,56} \xrightarrow{1} \mathcal{A}_{0,57}$ ),  $h = 1, r = 7, \mathbb{R} \xrightarrow{1} \mathbb{C}$ ;
- 2)  $\mathcal{A}_{0,58}^+ \xrightarrow{2} \mathcal{A}_{0,58}$  ( $\mathcal{A}_{0,57} \xrightarrow{2} \mathcal{A}_{0,58}$ ),  $h = 2, r = 7, \mathbb{C} \xrightarrow{2} \mathbb{H}$ ;
- 3)  $\mathcal{A}_{0,59}^+ \xrightarrow{3} \mathcal{A}_{0,59}$  ( $\mathcal{A}_{0,58} \xrightarrow{3} \mathcal{A}_{0,59}$ ),  $h = 3, r = 7, \mathbb{H} \xrightarrow{3} \mathbb{H} \oplus \mathbb{H}$ ;
- 4)  $\mathcal{A}_{0,60}^+ \xrightarrow{4} \mathcal{A}_{0,60}$  ( $\mathcal{A}_{0,59} \xrightarrow{4} \mathcal{A}_{0,60}$ ),  $h = 4, r = 7, \mathbb{H} \oplus \mathbb{H} \xrightarrow{4} \mathbb{H}$ ;
- 5)  $\mathcal{A}_{0,61}^+ \xrightarrow{5} \mathcal{A}_{0,61}$  ( $\mathcal{A}_{0,60} \xrightarrow{5} \mathcal{A}_{0,61}$ ),  $h = 5, r = 7, \mathbb{H} \xrightarrow{5} \mathbb{C}$ ;
- 6)  $\mathcal{A}_{0,62}^+ \xrightarrow{6} \mathcal{A}_{0,62}$  ( $\mathcal{A}_{0,61} \xrightarrow{6} \mathcal{A}_{0,62}$ ),  $h = 6, r = 7, \mathbb{C} \xrightarrow{6} \mathbb{R}$ ;
- 7)  $\mathcal{A}_{0,63}^+ \xrightarrow{7} \mathcal{A}_{0,63}$  ( $\mathcal{A}_{0,62} \xrightarrow{7} \mathcal{A}_{0,63}$ ),  $h = 7, r = 7, \mathbb{R} \xrightarrow{7} \mathbb{R} \oplus \mathbb{R}$ ;
- 8)  $\mathcal{A}_{0,64}^+ \xrightarrow{8} \mathcal{A}_{0,64}$  ( $\mathcal{A}_{0,63} \xrightarrow{8} \mathcal{A}_{0,64}$ ),  $h = 8, r = 7, \mathbb{R} \oplus \mathbb{R} \xrightarrow{8} \mathbb{R}$ .



**Fig. 4:** The second cycle of  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$ .

It is obvious that a process, which leads us to the fractal self-similar algebraic structure of the second order (Fig. 5), can be continued to any order (to infinity). Therefore, we come here to a fractal self-similar structure which is analogous to a Sierpiński carpet [32]. The fractal dimension (Besicovitch-Hausdorff dimension) of this structure is equal to  $D = \ln 63 / \ln 8 \approx 1,9924$ .

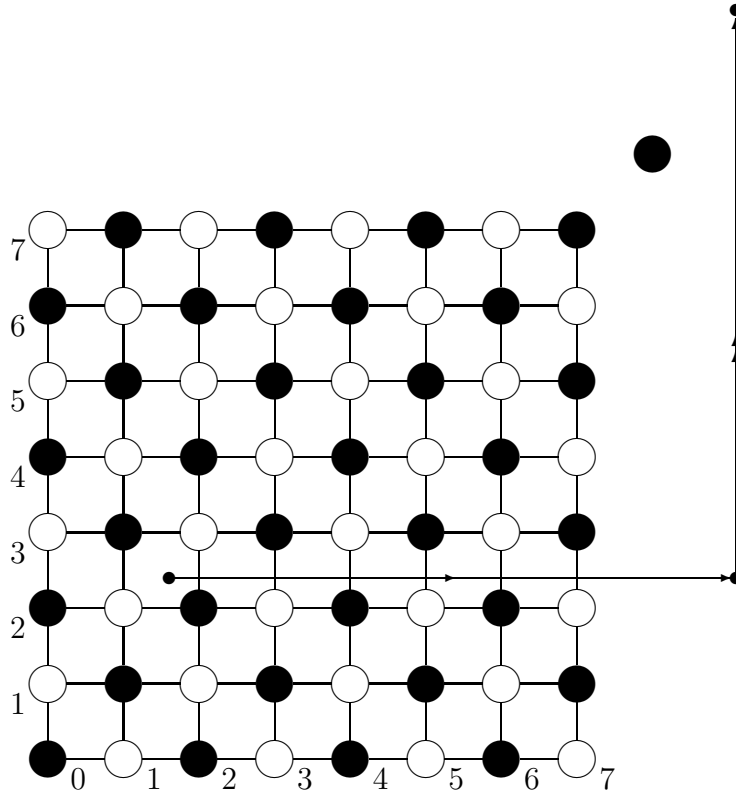
**Theorem 3.** *Over the field  $\mathbb{F} = \mathbb{R}$  the action of the Brauer-Wall group  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$  induces modulo 4 periodic relations for the idempotent group  $T_{p,q}(f) \simeq (\mathbb{Z}_2)^{\otimes(k+1)}$ :*

$$(\mathbb{Z}_2)^{\otimes(1+q-r_{q-p}+4)} \simeq (\mathbb{Z}_2)^{\otimes(1+q-r_{q-p})} \otimes (\mathbb{Z}_2)^{\otimes 4}.$$

*Proof.* Let us consider several cycles of  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$  on the set of the groups  $(\mathbf{e}_{\alpha_1}, \mathbf{e}_{\alpha_2}, \dots, \mathbf{e}_{\alpha_k}) \simeq (\mathbb{Z}_2)^{\otimes k}$ . The first step  $\mathcal{Cl}_{0,0} \xrightarrow{1} \mathcal{Cl}_{0,1}$  of the first cycle ( $h = 1, r = 1$ ) induces a transition  $(\mathbb{Z}_2)^0 \xrightarrow{1} (\mathbb{Z}_2)^0$ , since  $k = q - r_{q-p} = 0 - r_0 = 0$  ( $\mathcal{Cl}_{0,0}$ ) and  $k = q - r_{q-p} = 1 - r_1 = 0$  ( $\mathcal{Cl}_{0,1}$ ). The second step  $\mathcal{Cl}_{0,1} \xrightarrow{2} \mathcal{Cl}_{0,2}$  ( $h = 2, r = 1$ ) gives also  $(\mathbb{Z}_2)^0 \xrightarrow{1} (\mathbb{Z}_2)^0$ , since  $k = 2 - r_2 = 0$  for the algebra  $\mathcal{Cl}_{0,2}$ . The third step  $\mathcal{Cl}_{0,2} \xrightarrow{3} \mathcal{Cl}_{0,3}$  of the first cycle ( $h = 3, r = 1$ ) induces a transition  $(\mathbb{Z}_2)^0 \xrightarrow{3} (\mathbb{Z}_2)^{\otimes 1}$  ( $1 \xrightarrow{3} \mathbb{Z}_2$ ), since in this case we have  $k = 3 - r_3 = 1$  for  $\mathcal{Cl}_{0,3}$ . The following step  $\mathcal{Cl}_{0,3} \xrightarrow{4} \mathcal{Cl}_{0,4}$  ( $h = 4, r = 0$ ) of the group  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$  gives  $(\mathbb{Z}_2)^{\otimes 1} \xrightarrow{4} (\mathbb{Z}_2)^{\otimes 1}$  ( $\mathbb{Z}_2 \xrightarrow{4} \mathbb{Z}_2$ ) in virtue of  $k = 4 - r_4 = 1$  for  $\mathcal{Cl}_{0,4}$ . The fifth step  $\mathcal{Cl}_{0,4} \xrightarrow{5} \mathcal{Cl}_{0,5}$  induces a transition  $(\mathbb{Z}_2)^{\otimes 1} \xrightarrow{5} (\mathbb{Z}_2)^{\otimes 2}$  ( $\mathbb{Z}_2 \xrightarrow{5} \mathbb{Z}_2 \otimes \mathbb{Z}_2$ ), since  $k = 5 - r_5 = 2$  for  $\mathcal{Cl}_{0,5}$ . In turn, the sixth and seventh steps  $\mathcal{Cl}_{0,5} \xrightarrow{6} \mathcal{Cl}_{0,6}$  and  $\mathcal{Cl}_{0,6} \xrightarrow{7} \mathcal{Cl}_{0,7}$  of  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$  generate transitions  $(\mathbb{Z}_2)^{\otimes 2} \xrightarrow{6} (\mathbb{Z}_2)^{\otimes 3}$  ( $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \xrightarrow{6} \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$ ) and  $(\mathbb{Z}_2)^{\otimes 3} \xrightarrow{7} (\mathbb{Z}_2)^{\otimes 4}$  ( $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \xrightarrow{7} \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_4$ ), since  $k = 6 - r_6 = 3$  ( $\mathcal{Cl}_{0,6}$ ) and  $k = 7 - r_7 = 4$  ( $\mathcal{Cl}_{0,7}$ ). Finally, the eighth step  $\mathcal{Cl}_{0,7} \xrightarrow{8} \mathcal{Cl}_{0,8}$  induces a transition  $(\mathbb{Z}_2)^{\otimes 4} \xrightarrow{8} (\mathbb{Z}_2)^{\otimes 4}$ , since in virtue of the modulo 8 periodic relation  $r_{i+8} = r_i + 4$  for the Radon-Hurwitz numbers we have  $k = q - r_{q-p} = 8 - r_8 = 4 - r_0 = 4$  ( $\mathcal{Cl}_{0,8}$ ).

The second cycle ( $r = 1$ ) is started by the step  $\mathcal{Cl}_{0,8} \xrightarrow{1} \mathcal{Cl}_{0,9}$  which leads to  $(\mathbb{Z}_2)^{\otimes 4} \xrightarrow{1} (\mathbb{Z}_2)^{\otimes 4}$  and so on:

- 1)  $h = 1, r = 1, \mathcal{Cl}_{0,8} \xrightarrow{1} \mathcal{Cl}_{0,9} \mapsto (\mathbb{Z}_2)^{\otimes 4} \xrightarrow{1} (\mathbb{Z}_2)^{\otimes 4}, k = 9 - r_9 = 5 - r_1 = 4;$
- 2)  $h = 2, r = 1, \mathcal{Cl}_{0,9} \xrightarrow{2} \mathcal{Cl}_{0,10} \mapsto (\mathbb{Z}_2)^{\otimes 4} \xrightarrow{2} (\mathbb{Z}_2)^{\otimes 4}, k = 10 - r_{10} = 6 - r_2 = 4;$
- 3)  $h = 3, r = 1, \mathcal{Cl}_{0,10} \xrightarrow{3} \mathcal{Cl}_{0,11} \mapsto (\mathbb{Z}_2)^{\otimes 4} \xrightarrow{3} (\mathbb{Z}_2)^{\otimes 5}, k = 11 - r_{11} = 7 - r_3 = 5;$
- 4)  $h = 4, r = 1, \mathcal{Cl}_{0,11} \xrightarrow{4} \mathcal{Cl}_{0,12} \mapsto (\mathbb{Z}_2)^{\otimes 5} \xrightarrow{4} (\mathbb{Z}_2)^{\otimes 5}, k = 12 - r_{12} = 8 - r_4 = 5;$
- 5)  $h = 5, r = 1, \mathcal{Cl}_{0,12} \xrightarrow{5} \mathcal{Cl}_{0,13} \mapsto (\mathbb{Z}_2)^{\otimes 5} \xrightarrow{5} (\mathbb{Z}_2)^{\otimes 6}, k = 13 - r_{13} = 9 - r_5 = 6;$
- 6)  $h = 6, r = 1, \mathcal{Cl}_{0,13} \xrightarrow{6} \mathcal{Cl}_{0,14} \mapsto (\mathbb{Z}_2)^{\otimes 6} \xrightarrow{6} (\mathbb{Z}_2)^{\otimes 7}, k = 14 - r_{14} = 10 - r_6 = 7;$



**Fig. 5: The Spinorial Chessboard of the second order.** Black and white circles (squares of the board) present spinorial chessboards of the first order which are shown on the Fig. 1. These chessboards are distinguished from each other by the cycle number ( $r = 0, \dots, 7$ ) of the Brauer-Wall group  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$ .

$$7) \ h = 7, r = 1, \mathcal{O}_{0,14} \xrightarrow{7} \mathcal{O}_{0,15} \mapsto (\mathbb{Z}_2)^{\otimes 7} \xrightarrow{7} (\mathbb{Z}_2)^{\otimes 8}, k = 15 - r_{15} = 11 - r_7 = 8;$$

$$8) \ h = 8, r = 1, \mathcal{O}_{0,15} \xrightarrow{8} \mathcal{O}_{0,16} \mapsto (\mathbb{Z}_2)^{\otimes 8} \xrightarrow{8} (\mathbb{Z}_2)^{\otimes 8}, k = 16 - r_{16} = 8 - r_0 = 8.$$

In its turn, the third cycle of  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$  ( $r = 2$ ) is realized on the set of idempotent groups via the following eight steps:

$$1) \ h = 1, r = 2, \mathcal{O}_{0,16} \xrightarrow{1} \mathcal{O}_{0,17} \mapsto (\mathbb{Z}_2)^{\otimes 8} \xrightarrow{1} (\mathbb{Z}_2)^{\otimes 8}, k = 17 - r_{17} = 9 - r_1 = 8;$$

$$2) \ h = 2, r = 2, \mathcal{O}_{0,17} \xrightarrow{2} \mathcal{O}_{0,18} \mapsto (\mathbb{Z}_2)^{\otimes 8} \xrightarrow{2} (\mathbb{Z}_2)^{\otimes 8}, k = 18 - r_{18} = 10 - r_2 = 8;$$

$$3) \ h = 3, r = 2, \mathcal{O}_{0,18} \xrightarrow{3} \mathcal{O}_{0,19} \mapsto (\mathbb{Z}_2)^{\otimes 8} \xrightarrow{3} (\mathbb{Z}_2)^{\otimes 9}, k = 19 - r_{19} = 11 - r_3 = 9;$$

$$4) \ h = 4, r = 2, \mathcal{O}_{0,19} \xrightarrow{4} \mathcal{O}_{0,20} \mapsto (\mathbb{Z}_2)^{\otimes 9} \xrightarrow{4} (\mathbb{Z}_2)^{\otimes 9}, k = 20 - r_{20} = 12 - r_4 = 9;$$

$$5) \ h = 5, r = 2, \mathcal{O}_{0,20} \xrightarrow{5} \mathcal{O}_{0,21} \mapsto (\mathbb{Z}_2)^{\otimes 9} \xrightarrow{5} (\mathbb{Z}_2)^{\otimes 10}, k = 21 - r_{21} = 13 - r_5 = 10;$$

$$6) \ h = 6, r = 2, \mathcal{O}_{0,21} \xrightarrow{6} \mathcal{O}_{0,22} \mapsto (\mathbb{Z}_2)^{\otimes 10} \xrightarrow{6} (\mathbb{Z}_2)^{\otimes 11}, k = 22 - r_{22} = 14 - r_6 = 11;$$

$$7) \ h = 7, r = 2, \mathcal{O}_{0,22} \xrightarrow{7} \mathcal{O}_{0,23} \mapsto (\mathbb{Z}_2)^{\otimes 11} \xrightarrow{7} (\mathbb{Z}_2)^{\otimes 12}, k = 23 - r_{23} = 15 - r_7 = 12;$$

$$8) \ h = 8, r = 2, \mathcal{O}_{0,23} \xrightarrow{8} \mathcal{O}_{0,24} \mapsto (\mathbb{Z}_2)^{\otimes 12} \xrightarrow{8} (\mathbb{Z}_2)^{\otimes 12}, k = 24 - r_{24} = 12 - r_0 = 12.$$

Hence it immediately follows that the each cycle of  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$  generates on the set of  $(\mathbf{e}_{\alpha_1}, \mathbf{e}_{\alpha_2}, \dots, \mathbf{e}_{\alpha_k}) \simeq (\mathbb{Z}_2)^{\otimes k}$  a modulo 4 periodic relation  $(\mathbb{Z}_2)^{\otimes(k+4)} \simeq (\mathbb{Z}_2)^{\otimes k} \otimes (\mathbb{Z}_2)^{\otimes 4}$ . Therefore, for the idempotent group  $T_{p,q}(f)$  we have

$$(\mathbb{Z}_2)^{\otimes(1+q-r_{q-p}+4)} \simeq (\mathbb{Z}_2)^{\otimes(1+q-r_{q-p})} \otimes (\mathbb{Z}_2)^{\otimes 4}.$$

□



## 2.1 Spinor groups and quaternionic factorizations

A universal covering  $\mathbf{Spin}_+(p, q)$  of the rotation group  $\mathrm{SO}_0(p, q)$  of the pseudo-Euclidean space  $\mathbb{R}^{p,q}$  is described in terms of even subalgebra  $\mathcal{C}\ell_{p,q}^+$ . In its turn, the subalgebra  $\mathcal{C}\ell_{p,q}^+$  admits the following isomorphisms:  $\mathcal{C}\ell_{p,q}^+ \simeq \mathcal{C}\ell_{q,p-1}$ ,  $\mathcal{C}\ell_{p,q}^+ \simeq \mathcal{C}\ell_{p,q-1}$ . At this point, for the algebras of general type  $\mathcal{C}\ell_{p,q}$  ( $p \neq q$ ) and  $\mathcal{C}\ell_{p,0}$  we have

$$\mathcal{C}\ell_{p,q}^+ \simeq \mathcal{C}\ell_{q,p-1}. \quad (6)$$

The isomorphism

$$\mathcal{C}\ell_{p,q}^+ \simeq \mathcal{C}\ell_{p,q-1} \quad (7)$$

takes place for the algebras of type  $\mathcal{C}\ell_{0,q}$  and  $\mathcal{C}\ell_{p,p}$  ( $p = q$ ). In dependence on dimension of the associated space  $\mathbb{R}^{p,q}$ , the subalgebras  $\mathcal{C}\ell_{p,q-1}$  and  $\mathcal{C}\ell_{q,p-1}$  have even dimensions ( $p + q - 1 \equiv 0 \pmod{2}$ ) or odd dimensions ( $p + q - 1 \equiv 1 \pmod{2}$ ).

When  $p+q-1 \equiv 1 \pmod{2}$  we have four types  $q-p+1 \equiv 1, 3, 5, 7 \pmod{8}$  ( $p-q+1 \equiv 1, 3, 5, 7 \pmod{8}$ ) of the subalgebras  $\mathcal{C}\ell_{q,p-1}$  ( $\mathcal{C}\ell_{p,q-1}$ ). In this case a center  $\mathbf{Z}_{q,p-1}$  ( $\mathbf{Z}_{p,q-1}$ ) of  $\mathcal{C}\ell_{q,p-1}$  ( $\mathcal{C}\ell_{p,q-1}$ ) consists of the unit  $\mathbf{e}_0$  and the volume element  $\omega = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_{p+q-1}$ . When  $\omega^2 = -1$  ( $q-p+1 \equiv 3, 7 \pmod{8}$ ) or  $p-q+1 \equiv 3, 7 \pmod{8}$ ) we arrive at the isomorphisms  $\mathcal{C}\ell_{q,p-1} \simeq \mathbb{C}_{q+p-2} \simeq \mathbb{C} \left( 2^{\frac{q+p-2}{2}} \right)$ ,  $\mathcal{C}\ell_{0,p-1} \simeq \mathbb{C}_{p-2} \simeq \mathbb{C} \left( 2^{\frac{p-2}{2}} \right)$  and  $\mathcal{C}\ell_{0,q-1} \simeq \mathbb{C}_{q-2} \simeq \mathbb{C} \left( 2^{\frac{q-2}{2}} \right)$ . The transition from  $\mathcal{C}\ell_{q,p-1} \rightarrow \mathbb{C}_{q+p-2}$ ,  $\mathcal{C}\ell_{0,p-1} \rightarrow \mathbb{C}_{p-2}$  and  $\mathcal{C}\ell_{0,q-1} \rightarrow \mathbb{C}_{q-2}$  can be represented as the transition from the real coordinates in  $\mathcal{C}\ell_{q,p-1}$ ,  $\mathcal{C}\ell_{0,p-1}$  and  $\mathcal{C}\ell_{0,q-1}$  to complex coordinates  $a + \omega b$  in  $\mathbb{C}_{q+p-2}$ ,  $\mathbb{C}_{p-2}$  and  $\mathbb{C}_{q-2}$ , where  $\omega = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_{p+q-1} \in \mathcal{C}\ell_{q,p-1}$ ,  $\omega = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_{p-1} \in \mathcal{C}\ell_{0,p-1}$  and  $\omega = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_{q-1} \in \mathcal{C}\ell_{0,q-1}$ . On the other hand, when  $\omega^2 = 1$  ( $q-p+1 \equiv 1, 5 \pmod{8}$ ) or  $p-q+1 \equiv 1, 5 \pmod{8}$ ) we arrive at the isomorphisms

$$\begin{aligned} \mathcal{C}\ell_{q,p-1} &\simeq \mathcal{C}\ell_{p-1,q-1} \oplus \mathcal{C}\ell_{p-1,q-1} \simeq \mathbb{R} \left( 2^{\frac{p+q-2}{2}} \right) \oplus \mathbb{R} \left( 2^{\frac{p+q-2}{2}} \right), \\ \mathcal{C}\ell_{0,p-1} &\simeq \mathcal{C}\ell_{0,p-2} \oplus \mathcal{C}\ell_{0,p-2} \simeq \mathbb{R} \left( 2^{\frac{p-2}{2}} \right) \oplus \mathbb{R} \left( 2^{\frac{p-2}{2}} \right), \\ \mathcal{C}\ell_{0,q-1} &\simeq \mathcal{C}\ell_{0,q-2} \oplus \mathcal{C}\ell_{0,q-2} \simeq \mathbb{R} \left( 2^{\frac{q-2}{2}} \right) \oplus \mathbb{R} \left( 2^{\frac{q-2}{2}} \right) \end{aligned}$$

if  $q-p+1 \equiv 1 \pmod{8}$  (or  $p-1 \equiv 1 \pmod{8}$  and  $q-1 \equiv 1 \pmod{8}$ ). When  $q-p+1 \equiv 5 \pmod{8}$  (or  $p-1 \equiv 5 \pmod{8}$  and  $q-1 \equiv 5 \pmod{8}$ ) we have

$$\begin{aligned} \mathcal{C}\ell_{q,p-1} &\simeq \mathcal{C}\ell_{p-1,q-1} \oplus \mathcal{C}\ell_{p-1,q-1} \simeq \mathbb{H} \left( 2^{\frac{p+q-4}{2}} \right) \oplus \mathbb{H} \left( 2^{\frac{p+q-4}{2}} \right), \\ \mathcal{C}\ell_{0,p-1} &\simeq \mathcal{C}\ell_{0,p-2} \oplus \mathcal{C}\ell_{0,p-2} \simeq \mathbb{H} \left( 2^{\frac{p-4}{2}} \right) \oplus \mathbb{H} \left( 2^{\frac{p-4}{2}} \right), \\ \mathcal{C}\ell_{0,q-1} &\simeq \mathcal{C}\ell_{0,q-2} \oplus \mathcal{C}\ell_{0,q-2} \simeq \mathbb{H} \left( 2^{\frac{q-4}{2}} \right) \oplus \mathbb{H} \left( 2^{\frac{q-4}{2}} \right). \end{aligned}$$

The transitions  $\mathcal{C}\ell_{q,p-1} \rightarrow {}^2\mathbb{R} \left( 2^{\frac{p+q-2}{2}} \right)$ ,  $\mathcal{C}\ell_{0,p-1} \rightarrow {}^2\mathbb{R} \left( 2^{\frac{p-2}{2}} \right)$ ,  $\mathcal{C}\ell_{0,q-1} \rightarrow {}^2\mathbb{R} \left( 2^{\frac{q-2}{2}} \right)$  and  $\mathcal{C}\ell_{q,p-1} \rightarrow {}^2\mathbb{H} \left( 2^{\frac{p+q-4}{2}} \right)$ ,  $\mathcal{C}\ell_{0,p-1} \rightarrow {}^2\mathbb{H} \left( 2^{\frac{p-4}{2}} \right)$ ,  $\mathcal{C}\ell_{0,q-1} \rightarrow {}^2\mathbb{H} \left( 2^{\frac{q-4}{2}} \right)$  can be represented as the transition from the real coordinates in  $\mathcal{C}\ell_{q,p-1}$ ,  $\mathcal{C}\ell_{0,p-1}$ ,  $\mathcal{C}\ell_{0,q-1}$  to double coordinates  $a + \omega b$  in  ${}^2\mathbb{R} \left( 2^{\frac{p+q-2}{2}} \right)$ ,  ${}^2\mathbb{R} \left( 2^{\frac{p-2}{2}} \right)$ ,  ${}^2\mathbb{R} \left( 2^{\frac{q-2}{2}} \right)$  and  ${}^2\mathbb{H} \left( 2^{\frac{p+q-4}{2}} \right)$ ,  ${}^2\mathbb{H} \left( 2^{\frac{p-4}{2}} \right)$ ,  ${}^2\mathbb{H} \left( 2^{\frac{q-4}{2}} \right)$ .

Further, when  $p+q-1 \equiv 0 \pmod{2}$  we have four types  $q-p+1 \equiv 0, 2, 4, 6 \pmod{8}$  ( $p-q+1 \equiv 0, 2, 4, 6 \pmod{8}$ ) of the subalgebras  $\mathcal{C}\ell_{q,p-1}$  ( $\mathcal{C}\ell_{p,q-1}$ ). Let  $p+q-1 \geq 4$  and let  $m = (p+q-1)/4$  be an integer number (this number is always integer, since  $p+q-1 \equiv 0 \pmod{2}$ ).

(mod 2)). Then, at  $i \leq 2m$  we have

$$\begin{aligned}\mathbf{e}_{12\dots 2m2m+k}\mathbf{e}_i &= (-1)^{2m+1-i}\sigma(i-l)\mathbf{e}_{12\dots i-1i+1\dots 2m2m+k}, \\ \mathbf{e}_i\mathbf{e}_{12\dots 2m2m+k} &= (-1)^{i-1}\sigma(i-l)\mathbf{e}_{12\dots i-1i+1\dots 2m2m+k}\end{aligned}$$

and, therefore, the condition of commutativity of elements  $\mathbf{e}_{12\dots 2m2m+k}$  and  $\mathbf{e}_i$  is  $2m+1-i \equiv i-1 \pmod{2}$ . Thus, the elements  $\mathbf{e}_{12\dots 2m2m+1}$  and  $\mathbf{e}_{12\dots 2m2m+2}$  commute with all the elements  $\mathbf{e}_i$  whose indices do not exceed  $2m$ . Therefore, a transition from the algebra  $\mathcal{C}_{q,p-1}$  to algebras  $\mathcal{C}_{q+2,p-1}$ ,  $\mathcal{C}_{q,p+1}$  and  $\mathcal{C}_{q+1,p}$  can be represented as a transition from the real coordinates in  $\mathcal{C}_{q,p-1}$  to coordinates of the form  $a+b\phi+c\psi+d\phi\psi$ , where  $\phi$  and  $\psi$  are additional basis elements  $\mathbf{e}_{12\dots 2m2m+1}$  and  $\mathbf{e}_{12\dots 2m2m+2}$ . The elements  $\mathbf{e}_{i_1i_2\dots i_k}\phi$  contain the index  $2m+1$  and do not contain the index  $2m+2$ . In its turn, the elements  $\mathbf{e}_{i_1i_2\dots i_k}\psi$  contain the index  $2m+2$  and do not contain the index  $2m+1$ . Correspondingly, the elements  $\mathbf{e}_{i_1i_2\dots i_k}\phi\psi$  contain both indices  $2m+1$  and  $2m+2$ . Analogously, we have a transition from the algebra  $\mathcal{C}_{0,p-1}$  to  $\mathcal{C}_{2,p-1}$ ,  $\mathcal{C}_{0,p+1}$  and  $\mathcal{C}_{1,p}$ , from the algebra  $\mathcal{C}_{0,q-1}$  to  $\mathcal{C}_{2,q-1}$ ,  $\mathcal{C}_{0,q+1}$  and  $\mathcal{C}_{1,q}$ , and from the algebra  $\mathcal{C}_{p,p-1}$  to  $\mathcal{C}_{p+2,p-1}$ ,  $\mathcal{C}_{p,p+1}$ ,  $\mathcal{C}_{p+1,p}$ . Hence it immediately follows that general elements of these algebras can be represented as

$$\mathcal{A}_{\mathcal{C}_{q+2,p-1}} = \mathcal{C}_{q,p-1}^0 \mathbf{e}_0 + \mathcal{C}_{q,p-1}^1 \phi + \mathcal{C}_{q,p-1}^2 \psi + \mathcal{C}_{q,p-1}^3 \phi\psi, \quad (8)$$

$$\mathcal{A}_{\mathcal{C}_{q,p+1}} = \mathcal{C}_{q,p-1}^0 \mathbf{e}_0 + \mathcal{C}_{q,p-1}^1 \phi + \mathcal{C}_{q,p-1}^2 \psi + \mathcal{C}_{q,p-1}^3 \phi\psi, \quad (9)$$

$$\mathcal{A}_{\mathcal{C}_{q+1,p}} = \mathcal{C}_{q,p-1}^0 \mathbf{e}_0 + \mathcal{C}_{q,p-1}^1 \phi + \mathcal{C}_{q,p-1}^2 \psi + \mathcal{C}_{q,p-1}^3 \phi\psi, \quad (10)$$

$$\mathcal{A}_{\mathcal{C}_{2,p-1}} = \mathcal{C}_{0,p-1}^0 \mathbf{e}_0 + \mathcal{C}_{0,p-1}^1 \phi + \mathcal{C}_{0,p-1}^2 \psi + \mathcal{C}_{0,p-1}^3 \phi\psi, \quad (11)$$

$$\mathcal{A}_{\mathcal{C}_{0,p+1}} = \mathcal{C}_{0,p-1}^0 \mathbf{e}_0 + \mathcal{C}_{0,p-1}^1 \phi + \mathcal{C}_{0,p-1}^2 \psi + \mathcal{C}_{0,p-1}^3 \phi\psi, \quad (12)$$

$$\mathcal{A}_{\mathcal{C}_{1,p}} = \mathcal{C}_{0,p-1}^0 \mathbf{e}_0 + \mathcal{C}_{0,p-1}^1 \phi + \mathcal{C}_{0,p-1}^2 \psi + \mathcal{C}_{0,p-1}^3 \phi\psi, \quad (13)$$

$$\mathcal{A}_{\mathcal{C}_{2,q-1}} = \mathcal{C}_{0,q-1}^0 \mathbf{e}_0 + \mathcal{C}_{0,q-1}^1 \phi + \mathcal{C}_{0,q-1}^2 \psi + \mathcal{C}_{0,q-1}^3 \phi\psi, \quad (14)$$

$$\mathcal{A}_{\mathcal{C}_{0,q+1}} = \mathcal{C}_{0,q-1}^0 \mathbf{e}_0 + \mathcal{C}_{0,q-1}^1 \phi + \mathcal{C}_{0,q-1}^2 \psi + \mathcal{C}_{0,q-1}^3 \phi\psi, \quad (15)$$

$$\mathcal{A}_{\mathcal{C}_{1,q}} = \mathcal{C}_{0,q-1}^0 \mathbf{e}_0 + \mathcal{C}_{0,q-1}^1 \phi + \mathcal{C}_{0,q-1}^2 \psi + \mathcal{C}_{0,q-1}^3 \phi\psi, \quad (16)$$

$$\mathcal{A}_{\mathcal{C}_{p+2,p-1}} = \mathcal{C}_{p,p-1}^0 \mathbf{e}_0 + \mathcal{C}_{p,p-1}^1 \phi + \mathcal{C}_{p,p-1}^2 \psi + \mathcal{C}_{p,p-1}^3 \phi\psi, \quad (17)$$

$$\mathcal{A}_{\mathcal{C}_{p,p+1}} = \mathcal{C}_{p,p-1}^0 \mathbf{e}_0 + \mathcal{C}_{p,p-1}^1 \phi + \mathcal{C}_{p,p-1}^2 \psi + \mathcal{C}_{p,p-1}^3 \phi\psi, \quad (18)$$

$$\mathcal{A}_{\mathcal{C}_{p+1,p}} = \mathcal{C}_{p,p-1}^0 \mathbf{e}_0 + \mathcal{C}_{p,p-1}^1 \phi + \mathcal{C}_{p,p-1}^2 \psi + \mathcal{C}_{p,p-1}^3 \phi\psi, \quad (19)$$

where  $\mathcal{C}_{q,p-1}^i$ ,  $\mathcal{C}_{0,p-1}^i$ ,  $\mathcal{C}_{0,q-1}^i$ ,  $\mathcal{C}_{p,p-1}^i$ , are the algebras with a general element  $\mathcal{A} = \sum_{k=0}^{2m} a^{i_1i_2\dots i_k} \mathbf{e}_{i_1i_2\dots i_k}$ . When the elements  $\phi = \mathbf{e}_{12\dots 2m2m+1}$  and  $\psi = \mathbf{e}_{12\dots 2m2m+2}$  satisfy the condition  $\phi^2 = \psi^2 = -1$  we see that the basis  $\{\mathbf{e}_0, \phi, \psi, \phi\psi\}$  is isomorphic to a basis of the quaternion algebra  $\mathcal{C}_{0,2}$ , therefore, the elements (9), (12), (15) and (18) are general elements of quaternionic algebras. In turn, when the elements  $\phi$  and  $\psi$  satisfy the condition  $\phi^2 = -\psi^2 = 1$ , the basis  $\{\mathbf{e}_0, \phi, \psi, \phi\psi\}$  is isomorphic to a basis of the anti-quaternion algebra  $\mathcal{C}_{1,1}$ , and the elements (10), (13), (16) and (19) are general elements of anti-quaternionic algebras. Further, when  $\phi^2 = \psi^2 = 1$  we have a basis of the pseudo-quaternion algebra  $\mathcal{C}_{2,0}$ , and the elements (8), (11), (14) and (17) are general elements of pseudo-quaternionic algebras.

Let us define matrix representations of the quaternion units  $\phi$  and  $\psi$  as follows:

$$\phi \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \psi \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Using these representations and (9), we obtain

$$\mathcal{C}_{p,q+1} \simeq \text{Mat}_2(\mathcal{C}_{p,q-1}) = \begin{bmatrix} \mathcal{C}_{p,q-1}^0 - i\mathcal{C}_{p,q-1}^3 & -\mathcal{C}_{p,q-1}^1 + i\mathcal{C}_{p,q-1}^2 \\ \mathcal{C}_{p,q-1}^1 + i\mathcal{C}_{p,q-1}^2 & \mathcal{C}_{p,q-1}^0 + i\mathcal{C}_{p,q-1}^3 \end{bmatrix}.$$

The analogous expression takes place for the algebra  $\mathcal{C}_{q,p+1}$  with the element (12) and so on.

For example, let us consider the algebra  $\mathcal{C}_{2,4}$  associated with a six-dimensional pseudo-Euclidean space  $\mathbb{R}^{2,4}$ . A universal covering  $\mathbf{Spin}_+(2,4)$  of the rotation group  $\text{SO}_0(2,4)$  of  $\mathbb{R}^{2,4}$  is described in terms of even subalgebra  $\mathcal{C}_{2,4}^+$ . The algebra  $\mathcal{C}_{2,4}$  has the type  $p - q \equiv 6 \pmod{8}$ , therefore, from (6) we have  $\mathcal{C}_{2,4}^+ \simeq \mathcal{C}_{4,1}$ , where  $\mathcal{C}_{4,1}$  is a de Sitter algebra associated with the space  $\mathbb{R}^{4,1}$ . In its turn,  $\mathcal{C}_{4,1}$  has the type  $p - q \equiv 3 \pmod{8}$  and, therefore, there is an isomorphism  $\mathcal{C}_{4,1} \simeq \mathbb{C}_4$ , where  $\mathbb{C}_4$  is a Dirac algebra. The algebra  $\mathbb{C}_4$  is a complexification of space-time algebra:  $\mathbb{C}_4 \simeq \mathbb{C} \otimes \mathcal{C}_{1,3}$ . Further, the space-time algebra  $\mathcal{C}_{1,3}$  admits the following factorization:  $\mathcal{C}_{1,3} \simeq \mathcal{C}_{1,1} \otimes \mathcal{C}_{0,2}$ . Hence it immediately follows that  $\mathcal{C}_{1,3} \simeq \mathbb{C} \otimes \mathcal{C}_{1,1} \otimes \mathcal{C}_{0,2}$ . Thus,

$$\mathbf{Spin}_+(2,4) = \{s \in \mathbb{C} \otimes \mathcal{C}_{1,1} \otimes \mathcal{C}_{0,2} \mid N(s) = 1\}. \quad (20)$$

On the other hand, in virtue of  $\mathcal{C}_{1,3} \simeq \mathcal{C}_{1,1} \otimes \mathcal{C}_{0,2}$  from (9) we have

$$\mathcal{A}_{\alpha_{1,3}} = \mathcal{C}_{1,1}^0 \mathbf{e}_0 + \mathcal{C}_{1,1}^1 \phi + \mathcal{C}_{1,1}^2 \psi + \mathcal{C}_{1,1}^3 \phi\psi,$$

where  $\phi = \mathbf{e}_{123}$ ,  $\psi = \mathbf{e}_{124}$ . Therefore,

$$\mathbf{Spin}_+(2,4) = \left\{ s \in \begin{bmatrix} \mathbb{C} \otimes \mathcal{C}_{1,1}^0 - i\mathbb{C} \otimes \mathcal{C}_{1,1}^3 & -\mathbb{C} \otimes \mathcal{C}_{1,1}^1 + i\mathbb{C} \otimes \mathcal{C}_{1,1}^2 \\ \mathbb{C} \otimes \mathcal{C}_{1,1}^1 + i\mathbb{C} \otimes \mathcal{C}_{1,1}^2 & \mathbb{C} \otimes \mathcal{C}_{1,1}^0 + i\mathbb{C} \otimes \mathcal{C}_{1,1}^3 \end{bmatrix} \mid N(s) = 1 \right\}. \quad (21)$$

### 3 Spinor structure and the group $\mathbf{Spin}_+(1,3)$

Any irreducible finite dimensional representation  $\tau_{li}$  of the group  $\text{SL}(2, \mathbb{C}) \simeq \mathbf{Spin}_+(1,3)$  corresponds to a **particle of the spin**  $s$ , where  $s = |l - \dot{l}|$  (see also [14, 10]). All the values of  $s$  are

$$-s, \quad -s + 1, \quad -s + 2, \quad \dots, \quad s$$

or

$$-|l - \dot{l}|, \quad -|l - \dot{l}| + 1, \quad -|l - \dot{l}| + 2, \quad \dots, \quad |l - \dot{l}|. \quad (22)$$

Here the numbers  $l$  and  $\dot{l}$  are

$$l = \frac{k}{2}, \quad \dot{l} = \frac{r}{2},$$

where  $k$  and  $r$  are factor quantities in the tensor product

$$\underbrace{\mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \dots \otimes \mathbb{C}_2}_{k \text{ times}} \bigotimes \underbrace{\mathbb{C}_2^* \otimes \mathbb{C}_2^* \otimes \dots \otimes \mathbb{C}_2^*}_{r \text{ times}} \quad (23)$$

associated with the representation  $\tau_{k/2, r/2}$  of  $\text{SL}(2, \mathbb{C})$ , where  $\mathbb{C}_2$  and complex conjugate  $\mathbb{C}_2^*$  are biquaternion algebras. In turn, a *spinspace*  $\mathcal{S}_{2k+r}$ , associated with the tensor product (23), is

$$\underbrace{\mathbb{S}_2 \otimes \mathbb{S}_2 \otimes \dots \otimes \mathbb{S}_2}_{k \text{ times}} \bigotimes \underbrace{\dot{\mathbb{S}}_2 \otimes \dot{\mathbb{S}}_2 \otimes \dots \otimes \dot{\mathbb{S}}_2}_{r \text{ times}}. \quad (24)$$

Usual definition of the spin we obtain at the restriction  $\tau_{li} \rightarrow \tau_{l,0}$  (or  $\tau_{li} \rightarrow \tau_{0,i}$ ), that is, at the restriction of  $\text{SL}(2, \mathbb{C})$  to its subgroup  $\text{SU}(2)$ . In this case the sequence of spin values (22) is reduced to  $-l, -l + 1, -l + 2, \dots, l$  (or  $-\dot{l}, -\dot{l} + 1, -\dot{l} + 2, \dots, \dot{l}$ ).

Let

$$\mathbf{S} = \mathbf{s}^{\alpha_1 \alpha_2 \dots \alpha_k \dot{\alpha}_1 \dot{\alpha}_2 \dots \dot{\alpha}_r} = \sum \mathbf{s}^{\alpha_1} \otimes \mathbf{s}^{\alpha_2} \otimes \dots \otimes \mathbf{s}^{\alpha_k} \otimes \mathbf{s}^{\dot{\alpha}_1} \otimes \mathbf{s}^{\dot{\alpha}_2} \otimes \dots \otimes \mathbf{s}^{\dot{\alpha}_r}$$

be a spintensor polynomial, then any pair of substitutions

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & k \\ \alpha_1 & \alpha_2 & \dots & \alpha_k \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & \dots & r \\ \dot{\alpha}_1 & \dot{\alpha}_2 & \dots & \dot{\alpha}_r \end{pmatrix}$$

defines a transformation  $(\alpha, \beta)$  mapping  $\mathbf{S}$  to the following polynomial:

$$P_{\alpha\beta} \mathbf{S} = \mathbf{s}^{\alpha(\alpha_1) \alpha(\alpha_2) \dots \alpha(\alpha_k) \beta(\dot{\alpha}_1) \beta(\dot{\alpha}_2) \dots \beta(\dot{\alpha}_r)}.$$

The spintensor  $\mathbf{S}$  is called a *symmetric spintensor* if at any  $\alpha, \beta$  the equality

$$P_{\alpha\beta} \mathbf{S} = \mathbf{S}$$

holds. The space  $\text{Sym}_{(k,r)}$  of symmetric spintensors has the dimensionality

$$\dim \text{Sym}_{(k,r)} = (k+1)(r+1). \quad (25)$$

The dimensionality of  $\text{Sym}_{(k,r)}$  is called a *degree of the representation*  $\tau_{li}$  of the group  $\text{SL}(2, \mathbb{C})$ . It is easy to see that  $\text{SL}(2, \mathbb{C})$  has representations of **any degree** (in contrast to  $\text{SU}(3)$ ,  $\text{SU}(6)$  and other groups of internal symmetries, see [10]).

For the each  $A \in \text{SL}(2, \mathbb{C})$  we define a linear transformation of the spintensor  $\mathbf{s}$  via the formula

$$\mathbf{s}^{\alpha_1 \alpha_2 \dots \alpha_k \dot{\alpha}_1 \dot{\alpha}_2 \dots \dot{\alpha}_r} \longrightarrow \sum_{(\beta)(\dot{\beta})} A^{\alpha_1 \beta_1} A^{\alpha_2 \beta_2} \dots A^{\alpha_k \beta_k} \overline{A}^{\dot{\alpha}_1 \dot{\beta}_1} \overline{A}^{\dot{\alpha}_2 \dot{\beta}_2} \dots \overline{A}^{\dot{\alpha}_r \dot{\beta}_r} \mathbf{s}^{\beta_1 \beta_2 \dots \beta_k \dot{\beta}_1 \dot{\beta}_2 \dots \dot{\beta}_r},$$

where the symbols  $(\beta)$  and  $(\dot{\beta})$  mean  $\beta_1, \beta_2, \dots, \beta_k$  and  $\dot{\beta}_1, \dot{\beta}_2, \dots, \dot{\beta}_r$ . This representation of  $\text{SL}(2, \mathbb{C})$  we denote as  $\tau_{\frac{k}{2}, \frac{r}{2}} = \tau_{li}$ . The each *irreducible* finite dimensional representation of  $\text{SL}(2, \mathbb{C})$  is equivalent to one from  $\tau_{k/2, r/2}$ .

All the representations  $\tau_{li}$  can be grouped into spin multiplets in the Hilbert space  $\mathbf{H}_{2s+1}^S \otimes \mathbf{H}_\infty$  (see Fig. 6).  $\mathbf{H}_{2s+1}^S \otimes \mathbf{H}_\infty$  is a subspace of the more general spin-charge Hilbert space  $\mathbf{H}^S \otimes \mathbf{H}^Q \otimes \mathbf{H}_\infty$  [10]. The vertical lines (*spin lines*) correspond to particles of the same spin (but different masses). The horizontal lines (*spin chains* or spin multiplets) correspond to particles of the same mass (but different spins). Along the each spin chain the numbers  $l$  and  $\dot{l}$  are changed as

$$l, l + \frac{1}{2}, l + 1, l + \frac{3}{2}, \dots, \dot{l},$$

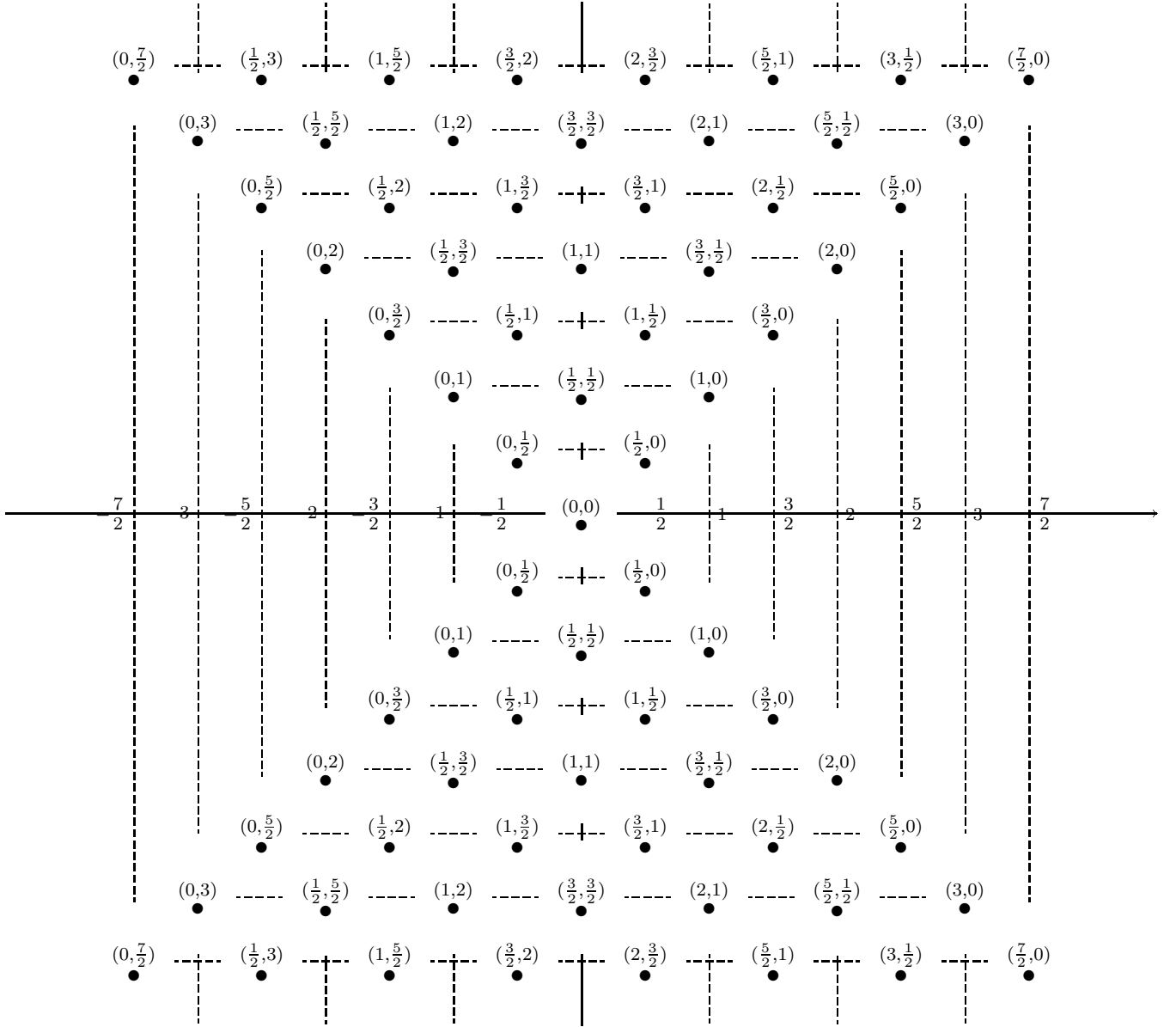
$$\dot{l}, \dot{l} - \frac{1}{2}, \dot{l} - 1, \dot{l} - \frac{3}{2}, \dots, l.$$

Therefore, along the each spin chain we have the following representations:

$$\tau_{li}, \tau_{l+\frac{1}{2}, \dot{l}-\frac{1}{2}}, \tau_{l+1, \dot{l}-1}, \tau_{l+\frac{3}{2}, \dot{l}-\frac{3}{2}}, \dots, \tau_{li},$$

where the spin  $s = l - \dot{l}$  is changed as

$$l - \dot{l}, l - \dot{l} + 1, l - \dot{l} + 2, l - \dot{l} + 3, \dots, \dot{l} - l.$$



**Fig. 6:** Matter and antimatter spin multiplets in  $\mathbf{H}_{2s+1}^S \otimes \mathbf{H}_\infty$ .

For example, let us consider the following spin chain (7-plet):

$$\begin{array}{ccccccccccc}
 (0,3) & \text{-----} & (\frac{1}{2}, \frac{5}{2}) & \text{-----} & (1,2) & \text{-----} & (\frac{3}{2}, \frac{3}{2}) & \text{-----} & (2,1) & \text{-----} & (\frac{5}{2}, \frac{1}{2}) & \text{-----} & (3,0) \\
 \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet \\
 -3 & \text{-----} & -2 & \text{-----} & -1 & \text{-----} & 0 & \text{-----} & 1 & \text{-----} & 2 & \text{-----} & 3
 \end{array}$$

In the underlying spinor structure we have the following sequence of algebras associated with this 7-plet:

$$\begin{aligned}
 \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 &\longleftrightarrow \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \longleftrightarrow \\
 \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 &\longleftrightarrow \mathbb{C}_2 \otimes \mathbb{C}_3 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \longleftrightarrow \\
 \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 &\longleftrightarrow \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \longleftrightarrow \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \mathbb{C}_2.
 \end{aligned}$$

Wave equations for the fields of type  $(l, 0) \oplus (0, l)$  and their solutions in the form of series in hyperspherical functions were given in [33]-[37]. It should be noted that  $(l, 0) \oplus (0, l)$  type wave

equations correspond to the usual definition of the spin. In turn, wave equations for the fields of type  $(l, \dot{l}) \oplus (\dot{l}, l)$  (arbitrary spin chains) and their solutions in the form of series in generalized hyperspherical functions were studied in [38]. Wave equations for arbitrary spin chains (spin multiplets) correspond to the generalized spin  $s = |l - \dot{l}|$ .

### 3.1 Spinor and twistor structures

The products (23) and (24) define an *algebraic (spinor) structure* associated with the representation  $\tau_{k/2, r/2}$  of the group  $\text{SL}(2, \mathbb{C})$ . Usually, spinor structures are understood as double (universal) coverings of the orthogonal groups  $\text{SO}(p, q)$ . For that reason it seems that the spinor structure presents itself a derivative construction. However, in accordance with Penrose twistor programme [5, 39, 40] the spinor (twistor) structure presents a more fundamental level of reality rather than space-time continuum. Moreover, space-time continuum is generated by the twistor structure. This is a natural consequence of the well known fact of the van der Waerden 2-spinor formalism [41], in which any vector of the Minkowski space-time can be constructed via the pair of mutually conjugated 2-spinors. For that reason it is more adequate to consider spinors as the *underlying structure*. We choose  $\mathbf{Spin}_+(1, 3)$  as a *generating kernel* of the underlying spinor structure. In this context space-time discrete symmetries  $P$ ,  $T$  and their combination  $PT$  should be considered as projections of the fundamental automorphisms belonging to the background spinor structure [13, 42, 43]. However, the group  $\mathbf{Spin}_+(2, 4) \simeq \text{SU}(2, 2)$  (a universal covering of the conformal group  $\text{SO}_0(2, 4)$ ) can be chosen as such a kernel. The choice  $\mathbf{Spin}_+(2, 4) \simeq \text{SU}(2, 2)$  takes place in the Penrose twistor programme [40] and also in the Paneitz-Segal approach [44]–[48].

#### 3.1.1 Spinors

Let us consider in brief the basic facts concerning the theory of spinor representations of the Lorentz group. The initial point of this theory is a correspondence between transformations of the proper Lorentz group and complex matrices of the second order. Indeed, following to [49] let us compare the Hermitian matrix of the second order

$$X = \begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix} \quad (26)$$

to the vector  $v$  of the Minkowski space-time  $\mathbb{R}^{1,3}$  with coordinates  $x_0, x_1, x_2, x_3$ . At this point,  $\det X = x_0^2 - x_1^2 - x_2^2 - x_3^2 = S^2(x)$ . The correspondence between matrices  $X$  and vectors  $v$  is one-to-one and linear. Any linear transformation  $X' = aXa^*$  in a space of the matrices  $X$  may be considered as a linear transformation  $g_a$  in  $\mathbb{R}^{1,3}$ , where  $a$  is a complex matrix of the second order with  $\det a = 1$ . The correspondence  $a \sim g_a$  possesses following properties: 1)  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim e$  (identity element); 2)  $g_{a_1}g_{a_2} = g_{a_1a_2}$  (composition); 3) two different matrices  $a_1$  and  $a_2$  correspond to one and the same transformation  $g_{a_1} = g_{a_2}$  only in the case  $a_1 = -a_2$ . Since the each complex matrix is defined by eight real numbers, then from the requirement  $\det a = 1$  it follow two conditions  $\text{Re } \det a = 1$  and  $\text{Im } \det a = 0$ . These conditions leave six independent parameters, that coincides with parameter number of the proper Lorentz group.

Further, a set of all complex matrices of the second order forms a full matrix algebra  $\text{Mat}_2(\mathbb{C})$  that is isomorphic to a biquaternion algebra  $\mathbb{C}_2$ . In turn, Pauli matrices

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (27)$$

form a spinbasis of the algebra  $\mathbb{C}_2$  (by this reason in physics the algebra  $\mathbb{C}_2 \simeq \mathcal{O}_{1,3}^+ \simeq \mathcal{O}_{3,0}$  is called Pauli algebra). Using the basis (27), we can write the matrix (26) in the form

$$X = x^\mu \sigma_\mu. \quad (28)$$

The Hermitian matrix (28) corresponds to a spintensor  $X^{\lambda\nu}$  with the following coordinates:

$$\begin{aligned} x^0 &= +(1/\sqrt{2})(\xi^1 \xi^{\dot{1}} + \xi^2 \xi^{\dot{2}}), & x^1 &= +(1/\sqrt{2})(\xi^1 \xi^{\dot{2}} + \xi^2 \xi^{\dot{1}}), \\ x^2 &= -(i/\sqrt{2})(\xi^1 \xi^{\dot{2}} - \xi^2 \xi^{\dot{1}}), & x^3 &= +(1/\sqrt{2})(\xi^1 \xi^{\dot{1}} - \xi^2 \xi^{\dot{2}}), \end{aligned} \quad (29)$$

where  $\xi^\mu$  and  $\xi^{\dot{\mu}}$  are correspondingly coordinates of spinors and cospinors of spinspace  $\mathbb{S}_2$  and  $\dot{\mathbb{S}}_2$ . Linear transformations of ‘vectors’ (spinors and cospinors) of the spinspace  $\mathbb{S}_2$  and  $\dot{\mathbb{S}}_2$  have the form

$$\begin{aligned} \begin{aligned} \xi'^1 &= \alpha \xi^1 + \beta \xi^2, \\ \xi'^2 &= \gamma \xi^1 + \delta \xi^2, \end{aligned} & \begin{aligned} \xi'^{\dot{1}} &= \dot{\alpha} \xi^{\dot{1}} + \dot{\beta} \xi^{\dot{2}}, \\ \xi'^{\dot{2}} &= \dot{\gamma} \xi^{\dot{1}} + \dot{\delta} \xi^{\dot{2}}, \end{aligned} \\ \sigma &= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} & \dot{\sigma} &= \begin{bmatrix} \dot{\alpha} & \dot{\beta} \\ \dot{\gamma} & \dot{\delta} \end{bmatrix}. \end{aligned} \quad (30)$$

Transformations (30) form the group  $\text{SL}(2, \mathbb{C})$ , since  $\sigma \in \text{Mat}_2(\mathbb{C})$  and

$$\text{SL}(2, \mathbb{C}) = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathbb{C}_2 : \det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = 1 \right\} \simeq \mathbf{Spin}_+(1, 3).$$

The expressions (29) and (30) compose a base of the 2-spinor van der Waerden formalism [3, 50], in which the spaces  $\mathbb{S}_2$  and  $\dot{\mathbb{S}}_2$  are called correspondingly spaces of *undotted and dotted spinors*. The each of the spaces  $\mathbb{S}_2$  and  $\dot{\mathbb{S}}_2$  is homeomorphic to an extended complex plane  $\mathbb{C} \cup \infty$  representing an absolute (a set of infinitely distant points) of a Lobatchevskii space  $S^{1,2}$ . At this point, a group of fractional linear transformations of the plane  $\mathbb{C} \cup \infty$  is isomorphic to a motion group of  $S^{1,2}$  [51]. Besides, the Lobatchevskii space  $S^{1,2}$  is an absolute of the Minkowski world  $\mathbb{R}^{1,3}$  and, therefore, the group of fractional linear transformations of the plane  $\mathbb{C} \cup \infty$  (motion group of  $S^{1,2}$ ) twice covers a ‘rotation group’ of the space-time  $\mathbb{R}^{1,3}$ , that is, the proper Lorentz group.

### 3.1.2 Twistors

The main idea of the Penrose twistor programme lies in the understanding of classical space-time as a some secondary construction which should be derived from the more primary notions. In capacity of the more primary notions we have here 2-component (complex) spinors, moreover, the pairs of 2-component spinors. In Penrose programme they called *twistors*. It is interesting to note that twistor theory gives a mathematical description of physics which based totally on the complex structure. At this point, space-time geometry and quantum mechanical superposition principle arise as closely related aspects of this complex twistor structure.

Twistor  $\mathbf{Z}^\alpha$  is constructed by the pair of 2-component quantities: spinor  $\omega^s$  and covariant spinor  $\pi_{\dot{s}}$  from conjugated space, that is,  $\mathbf{Z}^\alpha = (\omega^s, \pi_{\dot{s}})$  (or  $\mathbf{Z}^\alpha = (\xi^\mu, \xi_{\dot{\mu}})$ ). In twistor theory momentum ( $\vec{\omega}$ ) and impulse ( $\vec{\pi}$ ) of the particle are constructed from the quantities  $\omega^s$  and  $\pi_{\dot{s}}$ . One of the most important moments of this theory is *a transition from twistors to coordinate space-time*. Penrose described this transition with the help of so-called *basic relation of twistor theory*

$$\omega^s = i x^{s\dot{r}} \pi_{\dot{s}}, \quad (31)$$

where  $x^{s\dot{r}}$  is a mixed spintensor of the second rank (see (26)). In more details we have

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \frac{i}{\sqrt{2}} \begin{bmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix} \begin{bmatrix} \pi_{\dot{1}} \\ \pi_{\dot{2}} \end{bmatrix}.$$

From the basic relation (31) it immediately follows that space-time points are re-established over twistor space (they correspond to definite linear subspaces), but these points are secondary notion with respect to twistors.

In fact, twistors can be considered as ‘reduced spinors’<sup>1</sup> for a pseudo-unitary group  $SO_0(2, 4)$  which acts in six-dimensional space. This group is isomorphic locally to a 15-parameter conformal group of the Minkowski space  $\mathbb{R}^{1,3}$  (the group of point-to-point mappings of  $\mathbb{R}^{1,3}$  onto itself with preservation of the conformal structure of this space). Such mappings induce linear transformations of the twistor space which preserve the form  $\mathbf{Z}^\alpha \bar{\mathbf{Z}}_\alpha$ . The signature of  $\mathbf{Z}^\alpha \bar{\mathbf{Z}}_\alpha$  has the form  $(+, +, -, -)$ , it means that the corresponding group in the twistor space is  $SU(2, 2)$  (the group of pseudo-unitary  $(+, +, -, -)$  unimodular  $4 \times 4$  matrices, see also (21)):

$$SU(2, 2) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}_4 : \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 1 \right\} \simeq \mathbf{Spin}_+(2, 4).$$

### 3.1.3 Qubits

As is known, the qubit is a state vector of the two-level system. Thus, the qubit is a minimally possible (elementary) state vector. Any state vector can be represented as a set of such elementary vectors, for that reason the qubit is an original ‘building block’ for the all other state vectors of any dimension. The vector (qubit) of the two-level system can be written in the form

$$|\psi\rangle = a|0\rangle + b|1\rangle, \quad (32)$$

where  $a, b \in \mathbb{C}$ . A space of the two states, when the system can transits from one state to another (two-level system), is a simplest Hilbert space. The quantum state of  $N$  qubits can be expressed as a vector in a space of dimension  $2^N$ . It is obvious that this space coincides with the spinspace  $\mathbb{S}_{2N}$ . We can choose as an orthonormal basis for this space the states in which each qubit has a definite value, either  $|0\rangle$  or  $|1\rangle$ . These can be labeled by binary strings such as

$$|01110010 \dots 1001\rangle.$$

A general normalized vector can be expressed in this basis as  $\sum_{x=0}^{2^N-1} a_x |x\rangle$ , where  $a_x$  are complex numbers satisfying  $\sum_x |a_x|^2 = 1$ . Here we have a deep analogy between qubits and 2-component spinors. Just like the qubits, 2-component spinors are ‘building blocks’ of the underlying spinor structure (via the tensor products of  $\mathbb{C}_2$  and  $\mathbb{C}_2^*$ , see (23) and (24)). Moreover, vectors of the Hilbert space  $\mathbf{H}^S \otimes \mathbf{H}^Q \otimes \mathbf{H}_\infty$  are constructed via the same way [10], see also [52].

The density matrix of the qubit has  $2 \times 2$  size and for pure state (32) can be written as

$$\mathbf{r} = |\psi\rangle \langle\psi| = \begin{bmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{bmatrix}. \quad (33)$$

---

<sup>1</sup>These reduced spinors are understood as follows. General spinors are elements of the minimal left ideal of the conformal algebra  $\mathcal{O}_{2,4}$ ,

$$I_{2,4} = \mathcal{O}_{2,4} f_{24} = \mathcal{O}_{2,4} \frac{1}{2}(1 + \mathbf{e}_{15}) \frac{1}{2}(1 + \mathbf{e}_{26}).$$

The reduced spinors (twistors) are formulated within the even subalgebra  $\mathcal{O}_{2,4}^+ \simeq \mathcal{O}_{4,1}$  (the de Sitter algebra). The minimal left ideal of  $\mathcal{O}_{4,1} \simeq \mathbb{C}_4$  is (see also (1))

$$I_{4,1} = \mathcal{O}_{4,1} f_{4,1} = \mathcal{O}_{4,1} \frac{1}{2}(1 + \mathbf{e}_0) \frac{1}{2}(1 + i\mathbf{e}_{12}).$$

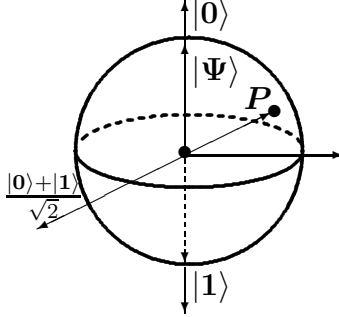
Therefore, after reduction  $I_{2,4} \rightarrow I_{4,1}$ , generated by the isomorphism  $\mathcal{O}_{2,4}^+ \simeq \mathcal{O}_{4,1}$ , we see that twistors  $\mathbf{Z}^\alpha$  are elements of the ideal  $I_{4,1}$  which leads to  $SU(2, 2) \simeq \mathbf{Spin}_+(2, 4) \in \mathcal{O}_{2,4}^+$  (see (20) and (21)). Moreover, from (2) we have a relation between twistors and Dirac-Hestenes spinors.



There exists a more general expression for the density matrix of the qubit which includes both pure and mixed states:

$$\mathbf{r} = \frac{1}{2}(\sigma_0 + \mathbf{P} \cdot \boldsymbol{\sigma}) = \frac{1}{2} \begin{bmatrix} 1 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & 1 - P_3 \end{bmatrix}, \quad (34)$$

where  $\mathbf{P} = (P_1, P_2, P_3)$  is a Bloch vector (polarization vector). Components of the Bloch vector are defined as average values of the Pauli matrices via the rule  $P_j = \langle \sigma_j \rangle = \text{Tr}(P_j \sigma_j)$ ,  $j = 1, 2, 3$ . In accordance with (34), three projections  $P_1, P_2, P_3$  of the polarization vector define the density matrix of the qubit. In the case of pure state the length of  $\mathbf{P}$  is equal to 1 ( $|\mathbf{P}|^2 = 1$ ) and this vector describes a sphere of the unit radius which called a *Bloch sphere* (see Fig. 7) In the case of



**Fig. 7: Bloch sphere.** The point  $P$  on the surface presents a pure state, and the points inside the ball present mixed states.

mixed state for the length of the polarization vector we have  $0 < |\mathbf{P}|^2 < 1$ . Therefore, the density matrix of the qubit can be represented by a point in three-dimensional space. That is, there exists one-to-one correspondence between the density matrix and the points of the unit ball. Pure states correspond to the points on the surface of this ball, and mixed states are described by the points inside the ball. Bloch sphere allows one to illustrate as a *classical domain* arises from a *quantum domain* in the process of decoherence [53]. When the system takes two possible values (positions) ‘up’ and ‘down’ along  $Z$ -axis, then in the limits of sphere the points on this axis present a totality of classical states, which can be appeared in the result of decoherence.

Carrying on the analogy between qubits and spinors (twistors), we see that there is a deep relationship between spinor structure (twistor programme) and theory of quantum information on the one hand and decoherence theory on the other hand. The underlying spinor structure presents by itself the mathematics that works on the level of nonlocal quantum substrate.

## 4 Modulo 8 periodicity and particle representations of $\text{Spin}_+(1, 3)$

**Theorem 4.** *The action of the group  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$  induces modulo 2 periodic relations on the system of real representations of the group  $\text{Spin}_+(1, 3) \simeq \text{SL}(2, \mathbb{C})$ .*

*Proof.* First of all, for the algebras of type  $\mathcal{C}_{0,q}$  ( $q \equiv 1 \pmod{2}$ ) there exists a decomposition  $\mathcal{C}_{0,q} \simeq \mathcal{C}_{0,q}^+ \oplus \mathcal{C}_{0,q}^-$ , where  $\mathcal{C}_{0,q}^+$  is an even subalgebra of  $\mathcal{C}_{0,q}$ . In virtue of an isomorphism  $\mathcal{C}_{0,q}^+ \simeq \mathcal{C}_{0,q-1}$  we have  $\mathcal{C}_{0,q} \simeq \mathcal{C}_{0,q-1} \oplus \mathcal{C}_{0,q-1}$ . This decomposition can be represented by the

following scheme:

$$\begin{array}{ccc} & \mathcal{C}_{0,q} & \\ \lambda_+ \swarrow & & \searrow \lambda_- \\ \mathcal{C}_{0,q-1} \oplus & & \mathcal{C}_{0,q-1} \end{array}$$

Here *central idempotents*

$$\lambda^+ = \frac{1 + \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_q}{2}, \quad \lambda^- = \frac{1 - \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_q}{2}$$

satisfy the relations  $(\lambda^+)^2 = \lambda^+$ ,  $(\lambda^-)^2 = \lambda^-$ ,  $\lambda^+ \lambda^- = 0$ . Further, there is a homomorphic mapping

$$\epsilon : \mathcal{C}_{0,q} \longrightarrow {}^\epsilon \mathcal{C}_{0,q-1}, \quad (35)$$

where

$${}^\epsilon \mathcal{C}_{0,q-1} \simeq \mathcal{C}_{0,q} / \text{Ker } \epsilon$$

is a quotient algebra,  $\text{Ker } \epsilon = \{\mathcal{A}^1 - \omega \mathcal{A}^1\}$  is a kernel of the homomorphism  $\epsilon$ ,  $\mathcal{A}^1 \in \mathcal{C}_{0,q-1}$  is an arbitrary element of  $\mathcal{C}_{0,q-1}$ , and  $\omega = \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_q \in \mathcal{C}_{0,q}$  is a volume element of  $\mathcal{C}_{0,q}$ . Therefore, in virtue of the homomorphic mapping (35) we can replace the double representations of  $\mathbf{Spin}_+(1, 3)$  by quotient representations  ${}^\epsilon \boldsymbol{\tau}_{0,0}^r$  and  ${}^\epsilon \boldsymbol{\tau}_{0,0}^q$ , where  ${}^\epsilon \boldsymbol{\tau}_{0,0}^r$  is a real quotient representation, and  ${}^\epsilon \boldsymbol{\tau}_{0,0}^q$  is a quaternionic quotient representation. About detailed structure of the quotient representations of  $\mathbf{Spin}_+(1, 3)$  see [11, 10].

On the underlying spinor structure the first step  $\mathcal{C}_{0,1}^+ \xrightarrow{1} \mathcal{C}_{0,1}$  ( $\mathcal{C}_{0,0} \xrightarrow{1} \mathcal{C}_{0,1}$ ) of the Brauer-Wall group  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$  generates a transition  $\boldsymbol{\tau}_{0,0}^r \xrightarrow{1} {}^\epsilon \boldsymbol{\tau}_{0,0}^r$ , where  $\boldsymbol{\tau}_{0,0}^r$  is a real representation of  $\mathbf{Spin}_+(1, 3)$  associated with the algebra  $\mathcal{C}_{0,0}$  ( $p - q \equiv 0 \pmod{8}$ ,  $\mathbb{K} \simeq \mathbb{R}$ ),  ${}^\epsilon \boldsymbol{\tau}_{0,0}^r$  is a real quotient representation of  $\mathbf{Spin}_+(1, 3)$  associated with the quotient algebra  ${}^\epsilon \mathcal{C}_{0,0} \simeq \mathcal{C}_{0,1} / \text{Ker } \epsilon$ , since in virtue of  $\mathcal{C}_{0,1}^+ \simeq \mathcal{C}_{0,0}$  we have  $\mathcal{C}_{0,1} \simeq \mathcal{C}_{0,0} \oplus i\mathcal{C}_{0,0}$ . The second step  $\mathcal{C}_{0,2}^+ \xrightarrow{2} \mathcal{C}_{0,2}$  ( $\mathcal{C}_{0,1} \xrightarrow{2} \mathcal{C}_{0,2}$ ) generates  ${}^\epsilon \boldsymbol{\tau}_{0,0}^r \xrightarrow{2} \boldsymbol{\tau}_{0,\frac{1}{2}}^q$ , where  $\boldsymbol{\tau}_{0,\frac{1}{2}}^q$  is a quaternionic representation of  $\mathbf{Spin}_+(1, 3)$  associated with the algebra  $\mathcal{C}_{0,2}$  ( $p - q \equiv 6 \pmod{8}$ ,  $\mathbb{K} \simeq \mathbb{H}$ ). The third step  $\mathcal{C}_{0,2} \xrightarrow{3} \mathcal{C}_{0,3}$  of  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$  induces a transition  $\boldsymbol{\tau}_{0,\frac{1}{2}}^q \xrightarrow{3} {}^\epsilon \boldsymbol{\tau}_{0,\frac{1}{2}}^q$ , where  ${}^\epsilon \boldsymbol{\tau}_{0,\frac{1}{2}}^q$  is a quaternionic quotient representation associated with the quotient algebra  ${}^\epsilon \mathcal{C}_{0,2} \simeq \mathcal{C}_{0,3} / \text{Ker } \epsilon$ . The following step  $\mathcal{C}_{0,3} \xrightarrow{4} \mathcal{C}_{0,4}$  generates  ${}^\epsilon \boldsymbol{\tau}_{0,\frac{1}{2}}^q \xrightarrow{4} \boldsymbol{\tau}_{0,1}^q$ , where  $\boldsymbol{\tau}_{0,1}^q$  is a quaternionic representation associated with  $\mathcal{C}_{0,4}$  ( $p - q \equiv 4 \pmod{8}$ ,  $\mathbb{K} \simeq \mathbb{H}$ ) in the underlying spinor structure. The fifth step  $\mathcal{C}_{0,4} \xrightarrow{5} \mathcal{C}_{0,5}$  of the first cycle of  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$  leads to  $\boldsymbol{\tau}_{0,1}^q \xrightarrow{5} {}^\epsilon \boldsymbol{\tau}_{0,1}^q$ , where  ${}^\epsilon \boldsymbol{\tau}_{0,1}^q$  is a quaternionic quotient representation associated with  ${}^\epsilon \mathcal{C}_{0,4} \simeq \mathcal{C}_{0,5} / \text{Ker } \epsilon$ . In turn, the sixth  $\mathcal{C}_{0,5} \xrightarrow{6} \mathcal{C}_{0,6}$  and seventh  $\mathcal{C}_{0,6} \xrightarrow{7} \mathcal{C}_{0,7}$  steps generate transitions  ${}^\epsilon \boldsymbol{\tau}_{0,1}^q \xrightarrow{6} \boldsymbol{\tau}_{0,\frac{3}{2}}^r$  and  $\boldsymbol{\tau}_{0,\frac{3}{2}}^r \xrightarrow{7} {}^\epsilon \boldsymbol{\tau}_{0,\frac{3}{2}}^r$ , where  $\boldsymbol{\tau}_{0,\frac{3}{2}}^r$  is a real representation of  $\mathbf{Spin}_+(1, 3)$  associated with the algebra  $\mathcal{C}_{0,6}$  ( $p - q \equiv 2 \pmod{8}$ ,  $\mathbb{K} \simeq \mathbb{R}$ ) and  ${}^\epsilon \boldsymbol{\tau}_{0,\frac{3}{2}}^r$  is a real quotient representation associated with  ${}^\epsilon \mathcal{C}_{0,6} \simeq \mathcal{C}_{0,7} / \text{Ker } \epsilon$ . The eighth step  $\mathcal{C}_{0,7} \xrightarrow{8} \mathcal{C}_{0,8}$  finishes the first cycle ( $r = 0$ ) of  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$  and induces a transition  ${}^\epsilon \boldsymbol{\tau}_{0,\frac{3}{2}}^r \xrightarrow{8} \boldsymbol{\tau}_{0,2}^r$ , where  $\boldsymbol{\tau}_{0,2}^r$  is a real representation associated with the algebra  $\mathcal{C}_{0,8}$ . The first cycle generates the first eight representations ( $\boldsymbol{\tau}_{0,0}^r, {}^\epsilon \boldsymbol{\tau}_{0,0}^r, \boldsymbol{\tau}_{0,\frac{1}{2}}^q, {}^\epsilon \boldsymbol{\tau}_{0,\frac{1}{2}}^q, \boldsymbol{\tau}_{0,1}^q, {}^\epsilon \boldsymbol{\tau}_{0,1}^q, \boldsymbol{\tau}_{0,\frac{3}{2}}^r, {}^\epsilon \boldsymbol{\tau}_{0,\frac{3}{2}}^r$ ) associated with the first eight squares ( $\mathcal{C}_{0,q}$ ,  $q = 0, \dots, 7$ ) of the spinorial chessboard (see Fig. 1). However, the pairs  $(\boldsymbol{\tau}_{0,0}^r, {}^\epsilon \boldsymbol{\tau}_{0,0}^r)$ ,  $(\boldsymbol{\tau}_{0,\frac{1}{2}}^q, {}^\epsilon \boldsymbol{\tau}_{0,\frac{1}{2}}^q)$ ,  $(\boldsymbol{\tau}_{0,1}^q, {}^\epsilon \boldsymbol{\tau}_{0,1}^q)$ ,  $(\boldsymbol{\tau}_{0,\frac{3}{2}}^r, {}^\epsilon \boldsymbol{\tau}_{0,\frac{3}{2}}^r)$  present particles of the same spin  $s$ , respectively,  $s = 0, \frac{1}{2}, 1, \frac{3}{2}$ .

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- 5)  $h = 5, r = 7, \mathcal{O}_{0,60} \xrightarrow{5} \mathcal{O}_{0,61} \rightsquigarrow \tau_{0,15}^q \xrightarrow{5} \epsilon \tau_{0,15}^q;$
- 6)  $h = 6, r = 7, \mathcal{O}_{0,61} \xrightarrow{6} \mathcal{O}_{0,62} \rightsquigarrow \epsilon \tau_{0,15}^q \xrightarrow{6} \tau_{0,\frac{31}{2}}^r;$
- 7)  $h = 7, r = 7, \mathcal{O}_{0,62} \xrightarrow{7} \mathcal{O}_{0,63} \rightsquigarrow \tau_{0,\frac{31}{2}}^r \xrightarrow{7} \epsilon \tau_{0,\frac{31}{2}}^r;$
- 8)  $h = 8, r = 7, \mathcal{O}_{0,63} \xrightarrow{8} \mathcal{O}_{0,64} \rightsquigarrow \epsilon \tau_{0,\frac{31}{2}}^r \xrightarrow{8} \tau_{0,16}^r,$

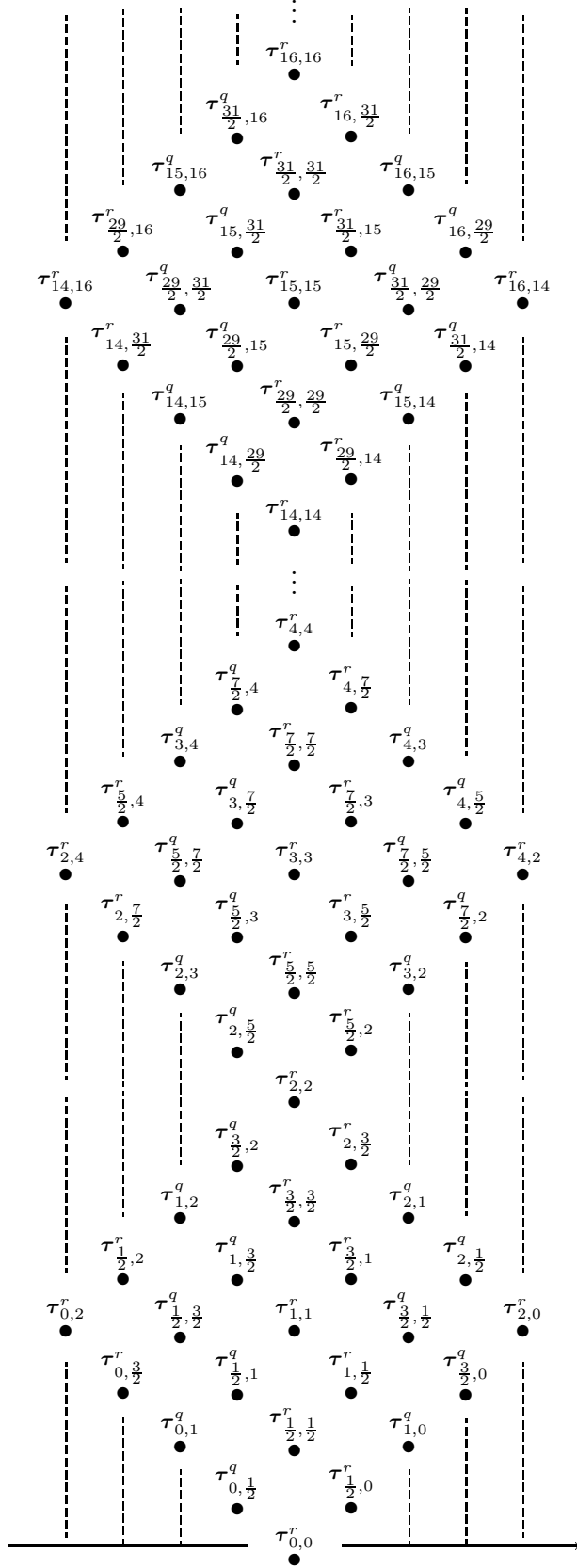
we come to a fractal self-similar algebraic structure of the second order which induces on the system of real representations of  $\mathbf{Spin}_+(1, 3)$  modulo 2 periodic structure shown on the Fig. 9. Thus, we have here the representation block of the second order, which, obviously, can be extended to infinity (to the blocks of any order) via the consecutive cycles of  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$ .  $\square$

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**Fig. 9:** The representation block of the second order of the group  $\mathbf{Spin}_+(1,3)$  (the main diagonal of this block). This block is generated by the eight cycles of the group  $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$ .